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Bayesian reciprocal bridge composite tobit quantile regression

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***Abstract*---**A composite tobit quantile regression approach is proposed for Bayesian simultaneous covariate selection and estimation in the setting of left censored regression. The proposed approach uses prior distributions for the regression coefficients that are scale mixtures of inverse uniform priors on the coefficients and independent Gamma priors on their mixing parameters. The proposed method was illustrated using simulation examples. Results show that the proposed method performs very well compared to the existing method.

***Keywords*---**composite quantile regression, posterior inference, left censored regression, regularization, Gibbs sampler, reciprocal bridge, tobit.

Introduction

High dimensional data studies arise across different fields of modern science such us economics, psychology, sociology, and signal processing. A considerable literature of variable selection methods has been proposed over the years in the context of linear mean regression from both frequentist and Bayesian framework. Regularization method is one of the variable selection methods that has proven successful for dealing with high-dimensional data. Bridge regression is a general class of the regularized mean regression approach introduced by Frank and Friedman (1993). However, the mean value is very sensitive to the contamination of the observations (Koenker and Bassett, 1978). Thus, if the data has a contamination in the observations, we expect the bridge regression to give us a poor prediction. Quantile regression (Koenker and Bassett, 1978), is a competitive alternative to mean regression for considering the relationship between the outcome of interest and covariates. By allowing varying nature across different quantile levels, quantile regression (QR) is able to accommodate heterogeneity and is robust to the contamination in the observations. Recently, Zou, and Yuan

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(2008) demonstrated that combining information over multiple quantiles (composite quantile) can improve the efficiency. Compared to the standard mean regression, Zou, and Yuan (2008) illustrated that the relative efficiency of composite QR across different distributions for the residuals is greater than 70%.

The advantage of composite QR can be used when a large number of observations on the outcome of interest assumes the value zero, i.e., ‘tobit models. This type of data is very popular and arise in a wide class of applications including ecology, economics, psychology, sociology, and signal processing. The tobit data simulating process can be written as follows:

$$y_i = \begin{cases} 0, & \text{if } y_i^* \leq 0; \\ y_i^*, & \text{if } y_i^* > 0; \end{cases} \tag{1}$$

where y_i is the observed outcome of ith observation determined by the latent unobserved response y_i^*. The linear tobit QR model for the θth quantile $(0 < \theta < 1)$ is $y^* = b_0 + X\beta + \varepsilon$, *where* $y = (y_1, \cdots, y_n)'$, $y^* = (y_1^*, \cdots, y_n^*)'$, $X = (x_1, \cdots, x_n)'$, b_0 is the intercept, $\beta = (\beta_1, \cdots, \beta_p)'$ and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)'$ are independent with their θth quantiles equal to 0. The θth tobit QR model takes the form of

$$Q_{y_i^*}(\theta \mid x_i) = b_\theta + x_i'\beta, \tag{2}$$

where b_θ is the tobit quantile intercept. Denote $0 < \theta_1 < \theta_2 < \cdots < \theta_K < 1$, where $\theta_k = k/(K+1)$.

Following Zou and Yuan (2008), the composite tobit QR estimators of $b_\theta = \left(b_{\theta_1}, \cdots, b_{\theta_K}\right)$ and β can be estimated by minimizing

$$\min_{b_{\theta_1}, \cdots, b_{\theta_K}, \beta} \sum_{k=1}^{K} \sum_{i=1}^{n} \rho_{\theta_k}\left(y_i - max\left\{0, b_{\theta_k} + x_i'\beta\right\}\right). \tag{3}$$

In this paper, we use an ALD-based working model (Yu and Stander, 2007) such that $y_i^* \sim \prod_{k=1}^{K} AL\left(b_{\theta_k} + x_i'\beta, \sigma, \theta_k\right)$, with a likelihood proportional given by

$$\prod_{k=1}^{K} \sigma^{-n} \exp\left\{-\sum_{i=1}^{n} \frac{\left|y_i - max\left\{0, b_{\theta_k} + \boldsymbol{x_i'\beta}\right\}\right| + (2\theta_k - 1)\left(y_i - max\left\{0, b_{\theta_k} + \boldsymbol{x_i'\beta}\right\}\right)}{2\sigma}\right\} \tag{4}$$

Similar to Alhamzawi and Yu (2013) and Kozumi and Kobayashi (2011), the check function (4) can be written as a scale mixture of normals as follows. For any a,m > 0, we have the following equality (Andrews and Mallows, 1974)

$$\exp\{-|am|\} = \int_0^\infty \frac{a}{\sqrt{2\pi v}} \exp\left\{-\frac{1}{2}\left(a^2 v + m^2 v^{-1}\right)\right\} dv \tag{5}$$

Letting $a = 1/\sqrt{2\sigma}, m = \varepsilon/\sqrt{2\sigma}$ and multiplying through by $exp\left\{-(2\theta_k - 1)\varepsilon/2\sigma\right\}$ gives

$$p(\boldsymbol{y} \mid \boldsymbol{X}, \boldsymbol{\beta}, \boldsymbol{b}_{\theta}, \boldsymbol{v}, \sigma) = \prod_{k=1}^{K}\prod_{i=1}^{n}\left(\frac{1}{\sqrt{4\pi\sigma v_{ik}}}\right)\exp\left\{-\frac{\left(y_i - max\left\{0, b_{\theta_k} + \boldsymbol{x}_i'\boldsymbol{\beta} + \xi_k v_{ik}\right\}\right)^2}{4\sigma v_{ik}}\right\} \qquad (6)$$

where $v = (v_1, \cdots, v_K)$ and $v_k = (v_{1k}, \cdots, v_{nk})$.
This representation has been used in (Alhamzawi and Ali (2018), Alhamzawi et al. (2011), Alshaybawee et al. (2017) , Yu et al. (2018))

Composite tobit QR with reciprocal bridge penalty

In this paper, we consider reciprocal bridge (rBridge) composite tobit QR (rBrCTQR) that results from the following regularization problem:

$$(\widehat{\boldsymbol{b}}_{\theta}, \widehat{\boldsymbol{\beta}}) = \min_{\boldsymbol{b}_{\theta}, \boldsymbol{\beta}} \sum_{i=1}^{n}\left\{\sum_{k=1}^{K} \rho_{\theta_k}\left(y_i - max\left\{0, b_{\theta_k} + \boldsymbol{x}_i'\boldsymbol{\beta} + \xi_k v_{ik}\right\}\right)\right\} + \lambda \sum_{j=1}^{p} \frac{1}{\left|\beta_j\right|^{\alpha}} I\{\beta_j \neq 0\}, \qquad (7)$$

where $b_{\theta} = \left(b_{\theta_1}, \cdots, b_{\theta_K}\right)$, $I(.)$ denotes an indicator function and $\lambda > 0$ is the tuning parameter that controls the degree of penalization. In this paper, rather than minimizing the above rBrCTQR problem in (7), we solve it by adopting a Bayesian hierarchical model and sampling the coefficients from its posterior distribution using a Gibbs sampler. The Bayesian analogue of the composite tobit QR with reciprocal bridge penalty (7) involves using an Inverse Generalized Gaussian (IGG) prior distribution of the form (Alhamzawi and Mallick, 2020a)

$$\pi(\beta) = \frac{\lambda^{\frac{1}{\alpha}}}{2\beta^2\Gamma\left(\frac{1}{\alpha}+1\right)} \exp\left\{-\frac{\lambda}{|\beta|^{\alpha}}\right\} I\{\beta \neq 0\}, \qquad (8)$$

where $\alpha > 0$ is a shape parameter and $\lambda > 0$ is a scale parameter. The prior (8) can be written as (Alhamzawi and Mallick, 2020a)

$$\frac{\lambda^{\frac{1}{\alpha}}}{2\beta^2\Gamma\left(\frac{1}{\alpha}+1\right)} e^{-\lambda|\beta|^{-\alpha}} = \frac{\lambda^{\frac{1}{\alpha}}}{2\beta^2\Gamma\left(\frac{1}{\alpha}+1\right)} \int_{u>|\beta|^{-\alpha}} \lambda e^{-\lambda u} du \qquad (9)$$

Hierarchical Bayesian Modelling

In this paper, following Mallick et al. (2021), Alhamzawi and Mallick (2020b) and Zainab and Alhamzawi (2022) we use the following hierarchical Bayesian modelling

$$y_i = \begin{cases} 0, & \text{if } y_i^* \leq 0; \\ y_i^*, & \text{if } y_i^* > 0; \end{cases}$$

$$y_i^* = \prod_{k=1}^{K}\left(b_{\theta k} + \boldsymbol{x}_i'\boldsymbol{\beta} + \xi_{1k} v_{ik} + \sqrt{\sigma \xi_{2k}^2 v_{ik}}\,\epsilon_i\right), i = 1, \cdots, n,$$

$$\boldsymbol{v} \mid \sigma \sim \prod_{k=1}^{K}\prod_{i=1}^{n} \sigma^{-1}\exp(\sigma^{-1} v_{ik}),$$

$$\epsilon \sim \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\epsilon_i^2}{2}\right), \qquad (10)$$

$$\beta \mid u, \alpha \sim \prod_{j=1}^{p} \frac{1}{\text{Uniform}\left(-u_j^{\frac{1}{\alpha}}, u_j^{\frac{1}{\alpha}}\right)},$$
$$u \mid \lambda, \alpha \sim \prod_{j=1}^{p} \text{Gamma}\left(\frac{1}{\alpha}+1, \lambda\right),$$
$$\lambda \sim \frac{\gamma^{\delta}}{\Gamma(\delta)} \lambda^{\delta-1} \exp\{-\gamma\lambda\},$$
$$\sigma \mid \sim 1/\sigma$$

Following Mallick et al. (2021) and Zainab and Alhamzawi (2022), the above Bayesian hierarchical modelling produce an efficient Gibbs sampler that works as follows.

Algorithm 1: MCMC sampling for the Bayesian bridge CQR.

- Input (Y, X)
- Initialize $(b_\theta, \beta, \sigma, v, u, \lambda, \alpha)$
- For $t = 1, \ldots, (t_{max} + t_{burn-in})$

1. Sample y_i^* as follows

$$y_i^* \mid y_i, b_\theta, \beta, v \sim \begin{cases} \Upsilon(y_i), & \text{if } y_i > 0 \\ N\left(b_{\theta_k} + x_i'\beta + \xi v_{ik}, 2\sigma v_{ik}\right) I(y_i^* \leq 0), & \text{otherwise,} \end{cases} \tag{11}$$

where Υ (.) denotes to a degenerate distribution.

2. Sample $\beta \mid . \sim N_p(\tilde{\beta}, \tilde{B}) \prod_{j=1}^{p} I\left\{|\tilde{\beta}_j| < u_j^{\frac{1}{\alpha}}\right\}$, where

$\tilde{B}^{-1} = \left(\sum_{i=1}^{n}\sum_{k=1}^{K} \frac{x_i x_i'}{2\sigma v_{ik}}\right)$ and

$\tilde{\beta} = \tilde{B}\left(\sum_{i=1}^{n}\sum_{k=1}^{K} \frac{x_i(y_i - b_{\theta k} - x_i'\beta - \xi_k v_{ik})}{2\sigma v_{ik}}\right)$

3. Sample $b_{\theta_k} \mid . \sim N\left(\frac{\sum_{i=1}^{n}(y_i - b_{\theta k} - x_i'\beta - \xi_k v_{ik})/2\sigma v_{ik}}{\sum_{i=1}^{n} 1/2\sigma v_{ik}}, \frac{1}{\sum_{i=1}^{n} 1/2\sigma v_{ik}}\right)$

4. Sample $v_{ik} \mid . \sim$ inverse Gaussian $\left(\frac{1}{2\sigma}, \sqrt{\frac{1}{\left(y_i - b_{\theta_k} - x_i'\beta\right)^2}}\right)$

5. Sample $\sigma \mid . \sim$ inverse Gamma $\left(\frac{3nK}{2}, \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{K} \frac{\left(y_i - b_{\theta k} - x_i'\beta - \xi_k v_{ik}\right)^2}{2v_{ik}} + \sum_{i=1}^{n}\sum_{k=1}^{K} \theta_k(1 - \theta_k)v_{ik}\right)$
6. Sample $u \mid . \sim \prod_{j=1}^{p}$ Exponential $(\lambda) I\{u_j > |\beta_j|^{\alpha}\}$

7. Sample $\lambda \mid . \sim Gamma\left(\delta + p/\alpha + p, \gamma + \sum_{j=1}^{p} |\beta_j|^{\alpha}\right)$
8. Sample $\gamma \mid . \sim (\delta, \lambda)$

9. Sample $\delta\,|.\sim(\gamma\lambda)^{p\delta}\Gamma(\delta)^{-p}$, which has no closed form. Since $p(\delta\,|.)$ is a log-concave, we update δ using Adaptive Rejection Sampling (ARS) (Gilks, 1992).

End For

Simulation studies

The performance of the proposed method was illustrated using simulation studies. The data in the simulation examples are generated by

$$y_i = \max\{0, y_i^*\},\ i = 1, \dots, n$$
$$y_i^* = x_i'\beta + \varepsilon_i,\ i = 1, \dots, n.$$

Three cases for the regression coefficient vector β were considered:

- Simulation 1: β = (2,2,0,0,2,0,0,0)
- Simulation 2: β = (2,00,0,0,0,0,0)
- Simulation 3: β = (1,1,1,1,1,1,1,1)

The rows of the design matrix X are generated from $N(0,\Sigma)$, where Σ has one of the following covariance structures:

1. Case I (IM): $\Sigma = I_p$ (Identity matrix of size p).
2. Case II (CS): $\Sigma_{ij} = 0.95$, if $i \neq j$ and $\Sigma_{ii} = 1$, for $i = 1, \dots, n$ (Compound symmetry matrix).
3. Case III (AR): $\Sigma_{ij} = 0.95^{|i-j|}$ for all $1 \leq i \leq j \leq p$ (Autoregressive correlated matrix).

Within each simulation study, we consider three choices for the error distribution ε: the normal distribution $N(0,9)$, $t_{(3)}$ and $\chi^2_{(3)}$ distribution. For each Case, we run 150 replications. In each replication, we simulate a training set with 30 observations and a testing set with 300 observations. We set a = 0.5 and K = 3. We use the credible interval criterion as suggested by Park and Casella (2008) to carry out variable selection.

Table 4.1: Estimation results for Simulation 1. All results are averaged over 100 replications and their associated standard deviations (SD) are listed in the parentheses. MMAD is the median of mean absolute deviations and RMSE is the relative mean square error. FPR is the false positive rate and FNR is the false negative rate.

				b_{θ_1}		b_{θ_2}		b_{θ_3}			
Method	Case	Error	MMAD SD	Bias	Rmse	Bias	Rmse	Bias	Rmse	FPR SD	FNR SD
crqL	I	N(0,9)	1.4644 (1.1033)	0.6459	0.7308	0.0637	0.3551	-0.8649	0.9491	0.4600 (0.2503)	0.0667 (0.1405)
BcrqL	I	N(0,9)	1.6995 (1.2787)	-2.7095	7.6063	0.1371	1.2884	3.1957	6.0752	0.2200 (0.1989)	0.1667 (0.2357)
BcrqB	I	N(0,9)	1.5055 (1.1595)	0.4856	0.5691	-0.0773	0.2247	-0.5219	0.6033	0.2600 (0.2119)	0.1667 (0.2357)
BcrqRB	I	N(0,9)	1.5129 (1.1658)	0.4827	0.5759	-0.0765	0.2307	-0.5329	0.6148	0.3200 (0.1932)	0.1333 (0.2331)
crqL	II	N(0,9)	1.4455 (1.0584)	0.5164	0.7372	-0.0192	0.3594	-0.6406	0.8045	1.000 (0.000)	0.000 (0.000)
BcrqL	II	N(0,9)	1.4552 (1.1061)	-1.2939	5.6695	-2.3873	8.2980	0.7129	2.2683	1.000 (0.000)	0.000 (0.000)

BcrqB	II	N(0,9)	1.5218 (1.1097)	0.4612	0.5255	-0.0512	0.2238	-0.5205	0.6585	1.000 (0.000)	0.000 (0.000)
BcrqRB	II	N(0,9)	1.5165 (1.1049)	0.4495	0.5188	-0.0496	0.2137	-0.5323	0.6478	1.000 (0.000)	0.000 (0.000)
crqL	III	N(0,9)	1.4718 (1.1156)	0.9783	1.0868	0.1324	0.5422	-0.7780	0.8698	0.8200 (0.2573)	0.000 (0.000)
BcrqL	III	N(0,9)	1.3824 (1.0242)	-0.1267	1.9774	0.9536	7.7891	2.8116	15.5061	0.8000 (0.2667)	0.000 (0.000)
BcrqB	III	N(0,9)	1.5432 (1.1308)	0.6586	0.7798	0.0843	0.2283	-0.5428	0.6218	0.7600 (0.3239)	0.000 (0.000)
BcrqRB	III	N(0,9)	1.5243 (1.1176)	0.6592	0.7822	0.1059	0.2426	-0.5425	0.6211	0.7600 (0.3239)	0.000 (0.000)
crqL	I	$t(3)$	1.4208 (1.0351)	0.1666	0.2724	-0.1121	0.1776	-0.2537	0.3552	0.2200 (0.2201)	0.000 (0.000)
BcrqL	I	$t(3)$	1.1234 (0.8746)	-0.7355	2.3399	0.2500	1.0798	1.4100	2.4647	0.1600 (0.1838)	0.000 (0.000)
BcrqB	I	$t(3)$	1.3623 (1.0272)	0.0433	0.1565	-0.0469	0.1396	-0.1126	0.1467	0.1000 (0.1054)	0.000 (0.000)
BcrqRB	I	$t(3)$	1.3706 (1.0241)	0.0514	0.1566	-0.0545	0.1572	-0.1121	0.1416	0.1400 (0.1647)	0.000 (0.000)
crqL	II	$t(3)$	1.2073 (0.8984)	0.1200	0.2694	-0.0938	0.1945	-0.1535	0.2638	0.9400 (0.1897)	0.000 (0.000)
BcrqL	II	$t(3)$	1.0349 (0.7644)	-0.9874	2.5891	1.0036	2.4529	1.0132	4.4328	0.9400 (0.1897)	0.000 (0.000)
BcrqB	II	$t(3)$	1.2506 (0.9584)	-0.0587	0.3176	-0.0107	0.1434	0.0515	0.1920	0.9000 (0.3162)	0.000 (0.000)
BcrqRB	II	$t(3)$	1.2609 (0.9540)	-0.0647	0.2988	-0.0108	0.1361	0.0495	0.1999	0.9200 (0.2530)	0.000 (0.000)
crqL	III	$t(3)$	0.9774 (0.7475)	0.1908	0.3006	-0.0572	0.2849	-0.0956	0.3690	0.7600 (0.3502)	0.000 (0.000)
BcrqL	III	$t(3)$	0.9134 (0.6842)	-4.5421	8.0489	0.6105	2.2257	2.9745	7.0045	0.7600 (0.3502)	0.000 (0.000)
BcrqB	III	$t(3)$	1.0764 (0.7747)	0.0494	0.2611	0.0137	0.1198	-0.0180	0.3530	0.7000 (0.4137)	0.000 (0.000)
BcrqRB	III	$t(3)$	1.0753 (0.7733)	0.0571	0.2467	0.0196	0.1138	-0.0090	0.3498	0.7400 (0.3777)	0.000 (0.000)
crqL	I	$\chi^2_{(3)}$	1.068 (0.8054)	-2.4725	2.5112	-2.6028	2.6338	-3.2973	3.3239	0.4600 (0.2319)	0.0333 (0.1054)
BcrqL	I	$\chi^2_{(3)}$	0.9585 (0.7114)	-3.939	5.9089	-0.4561	1.9956	3.2930	6.6940	0.3400 (0.2503)	0.0333 (0.1054)
BcrqB	I	$\chi^2_{(3)}$	1.1279 (0.8159)	-2.6078	2.6364	-2.6123	2.6406	-3.0812	3.0952	0.2800 (0.1932)	0.0333 (0.1054)
BcrqRB	I	$\chi^2_{(3)}$	1.1303 (0.8208)	-2.6124	2.6409	-2.6139	2.6445	-3.0808	3.0938	0.3200 (0.1687)	0.0333 (0.1054)
crqL	II	$\chi^2_{(3)}$	1.0883 (0.8102)	-2.2637	2.2858	-2.6213	2.6348	-3.4437	3.4589	1.000 (0.000)	0.000 (0.000)
BcrqL	II	$\chi^2_{(3)}$	0.9585 (0.7074)	-3.8331	7.4801	-1.7052	2.0696	1.1098	9.9763	1.000 (0.000)	0.000 (0.000)
BcrqB	II	$\chi^2_{(3)}$	1.1399 (0.8345)	-2.4584	2.4785	-2.5522	2.5612	-3.1241	3.1361	1.000 (0.000)	0.000 (0.000)
BcrqRB	II	$\chi^2_{(3)}$	1.1405 (0.8268)	-2.4588	2.4772	-2.5683	2.5779	-3.1291	3.1385	1.000 (0.000)	0.000 (0.000)
crqL	III	$\chi^2_{(3)}$	1.0883 (0.8102)	-2.4124	2.4411	-2.4888	2.5077	-3.1746	3.1992	0.7600 (0.3098)	0.0333 (0.1054)
BcrqL	III	$\chi^2_{(3)}$	0.9364 (0.6890)	-5.5112	8.7578	-1.2942	3.9786	0.3339	7.5585	0.7600 (0.3098)	0.0333 (0.1054)
BcrqB	III	$\chi^2_{(3)}$	1.1320 (0.8345)	-2.5263	2.5629	-2.4503	2.4569	-2.8984	2.9128	0.6800 (0.3676)	0.0333 (0.1054)
BcrqRB	III	$\chi^2_{(3)}$	1.1244 (0.8268)	-2.5070	2.5394	-2.4425	2.4490	-2.9010	2.9150	0.7200 (0.3553)	0.0333 (0.1054)

Table 4.2: Estimation results for Simulation 2. All results are averaged over 100 replications and their associated standard deviations (SD) are listed in the parentheses. MMAD is the median of mean absolute deviations and RMSE is the relative mean square error. FPR is the false positive rate and FNR is the false negative rate.

				b_{θ_1}		b_{θ_2}		b_{θ_3}			
Method	Case	Error	MMAD SD	Bias	Rmse	Bias	Rmse	Bias	Rmse	FPR SD	FNR SD
crqL	I	N(0,9)	1.1807 (0.8639)	0.8369	0.9241	0.0548	0.3591	-0.8486	0.9465	0.5000 (0.1543)	0.000 (0.000)
BcrqL	I	N(0,9)	0.9747 (0.7357)	-3.9665	12.286	-2.0305	4.6070	2.0191	5.5867	0.2857 (0.202)	0.100 (0.3162)
BcrqB	I	N(0,9)	1.1658 (0.8745)	0.5888	0.6387	0.0492	0.2290	-0.5640	0.6892	0.3143 (0.1881)	0.000 (0.000)
BcrqRB	I	N(0,9)	1.1667 (0.880)	0.5910	0.6408	0.0563	0.2637	-0.5673	0.6899	0.4714 (0.1355)	0.000 (0.000)
crqL	II	N(0,9)	1.2494 (0.9359)	1.0557	1.1414	0.0691	0.2558	-0.9590	1.0714	1.000 (0.000)	0.000 (0.000)
BcrqL	II	N(0,9)	1.0080 (0.7515)	0.1868	3.1962	0.0989	1.6025	1.6148	7.1906	1.000 (0.000)	0.000 (0.000)
BcrqB	II	N(0,9)	1.2331 (0.9211)	0.7370	0.7699	0.0280	0.2214	-0.7113	0.7944	1.000 (0.000)	0.000 (0.000)
BcrqRB	II	N(0,9)	1.2303 (0.9205)	0.7462	0.7819	0.0282	0.2171	-0.7036	0.7935	1.000 (0.000)	0.000 (0.000)

crqL	III	N(0,9)	1.2784 (0.9814)	0.8353	1.0057	0.0092	0.5456	-0.9591	1.1020	0.8429 (0.2279)	0.000 (0.000)
BcrqL	III	N(0,9)	0.9948 (0.7434)	0.2454	1.4225	0.1369	2.8113	3.2472	9.9302	0.6857 (0.3486)	0.000 (0.000)
BcrqB	III	N(0,9)	1.2697 (0.9671)	0.5785	0.6976	0.0320	0.3507	-0.6447	0.7829	0.6714 (0.3437)	0.000 (0.000)
BcrqRB	III	N(0,9)	1.2712 (0.9616)	0.5780	0.7029	0.0370	0.3422	-0.6481	0.7845	0.7429 (0.2409)	0.000 (0.000)
crqL	I	$t(3)$	1.2494 (0.9269)	-0.0295	0.3106	0.0141	0.1445	0.1347	0.3124	0.2429 (0.1911)	0.000 (0.000)
BcrqL	I	$t(3)$	0.9585 (0.7137)	-0.4638	0.7165	-0.0123	3.0884	3.3700	7.2501	0.0000 (0.000)	0.000 (0.000)
BcrqB	I	$t(3)$	1.2280 (0.9150)	-0.0846	0.2564	0.0159	0.1042	0.1153	0.2567	0.0000 (0.000)	0.000 (0.000)
BcrqRB	I	$t(3)$	1.2216 (0.9109)	-0.0929	0.2567	0.0224	0.1065	0.1159	0.2591	0.1714 (0.1622)	0.000 (0.000)
crqL	II	$t(3)$	1.1661 (0.8479)	0.0907	0.3934	-0.1287	0.3613	-0.2092	0.4050	1.000 (0.000)	0.000 (0.000)
BcrqL	II	$t(3)$	0.9141 (0.6886)	0.5801	4.9349	0.3528	1.6004	1.3934	3.0339	1.000 (0.000)	0.000 (0.000)
BcrqB	II	$t(3)$	1.1534 (0.8555)	-0.0636	0.3369	-0.0797	0.2768	-0.0760	0.2838	1.000 (0.000)	0.000 (0.000)
BcrqRB	II	$t(3)$	1.1612 (0.8543)	-0.0608	0.3409	-0.0650	0.2785	-0.0791	0.2946	1.000 (0.000)	0.000 (0.000)
crqL	III	$t(3)$	1.1296 (0.8255)	0.0428	0.2955	-0.0286	0.1403	-0.0880	0.4615	0.6571 (0.3381)	0.000 (0.000)
BcrqL	III	$t(3)$	0.8685 (0.6819)	-5.2089	11.1165	-0.2725	0.5146	1.3273	5.5820	0.5286 (0.4367)	0.000 (0.000)
BcrqB	III	$t(3)$	1.0951 (0.7967)	-0.0623	0.2496	-0.0166	0.1413	0.0314	0.3246	0.5143 (0.4477)	0.000 (0.000)
BcrqRB	III	$t(3)$	1.0922 (0.8101)	-0.0626	0.2373	-0.0201	0.1235	0.0319	0.3093	0.6143 (0.393)	0.000 (0.000)
crqL	I	$\chi^2_{(3)}$	1.1373 (0.8314)	-2.4246	2.4391	-2.5655	2.5883	-3.2330	3.2830	0.4143 (0.2279)	0.000 (0.000)
BcrqL	I	$\chi^2_{(3)}$	0.8571 (0.6482)	-2.8082	3.6069	-1.3664	5.0879	7.4835	23.3972	0.2000 (0.1928)	0.000 (0.000)
BcrqB	I	$\chi^2_{(3)}$	1.1023 (0.8143)	-2.5999	2.6243	-2.5589	2.5679	-3.0714	3.0914	0.2286 (0.1928)	0.000 (0.000)
BcrqRB	I	$\chi^2_{(3)}$	1.1036 (0.8126)	-2.6044	2.6282	-2.5583	2.5673	-3.0453	3.0663	0.3429 (0.1807)	0.000 (0.000)
crqL	II	$\chi^2_{(3)}$	1.1083 (0.8255)	-2.3200	2.3459	-2.8117	2.8262	-3.4371	3.4693	1.000 (0.000)	0.000 (0.000)
BcrqL	II	$\chi^2_{(3)}$	0.8420 (0.6238)	-2.7318	4.5963	-2.1081	2.3819	0.5747	4.2951	1.000 (0.000)	0.000 (0.000)
BcrqB	II	$\chi^2_{(3)}$	1.0826 (0.7944)	-2.4423	2.4585	-2.6377	2.6463	-3.2826	3.2889	1.000 (0.000)	0.000 (0.000)
BcrqRB	II	$\chi^2_{(3)}$	1.0866 (0.8082)	-2.4366	2.4539	-2.648	2.6569	-3.2883	3.2950	1.000 (0.000)	0.000 (0.000)
crqL	III	$\chi^2_{(3)}$	1.1296 (0.8314)	-2.0384	2.0568	-2.4253	2.4461	-3.2533	3.2727	0.9000 (0.2135)	0.000 (0.000)
BcrqL	III	$\chi^2_{(3)}$	0.8399 (0.6235)	-1.5523	2.2294	-2.1697	2.5568	-0.0587	4.4292	0.8143 (0.3930)	0.000 (0.000)
BcrqB	III	$\chi^2_{(3)}$	1.0826 (0.7944)	-2.3041	2.3166	-2.4562	2.4650	-3.1652	3.1772	0.8429 (0.3330)	0.000 (0.000)
BcrqRB	III	$\chi^2_{(3)}$	1.0866 (0.8082)	-2.3056	2.3182	-2.4568	2.4658	-3.1617	3.1733	0.8714 (0.2731)	0.000 (0.000)

Table 4.3: Estimation results for Simulation 3. All results are averaged over 100 replications and their associated standard deviations (SD) are listed in the parentheses. MMAD is the median of mean absolute deviations and RMSE is the relative mean square error. FPR is the false positive rate and FNR is the false negative rate.

				b_{θ_1}		b_{θ_2}		b_{θ_3}			
Method	Case	Error	MMAD SD	Bias	Rmse	Bias	Rmse	Bias	Rmse	FPR SD	FNR SD
crqL	I	N(0,9)	1.1544 (0.8543)	0.7517	0.8869	-0.1351	0.4594	-0.7833	0.9911	0.000 (0.000)	0.2125 (0.1672)
BcrqL	I	N(0,9)	0.8601 (0.6490)	0.6665	11.9104	1.2102	2.9099	5.4291	8.9329	0.000 (0.000)	0.3375 (0.2361)
BcrqB	I	N(0,9)	1.1201 (0.8345)	0.4741	0.5716	-0.0499	0.2634	-0.5817	0.7013	0.000 (0.000)	0.3500 (0.1845)
BcrqRB	I	N(0,9)	1.1184 (0.8360)	0.4634	0.5642	-0.0421	0.2517	-0.5624	0.6888	0.000 (0.000)	0.2750 (0.1419)
crqL	II	N(0,9)	1.1733 (0.8639)	0.8027	0.9151	0.1132	0.4747	-0.8573	0.9589	0.000 (0.000)	0.000 (0.000)
BcrqL	II	N(0,9)	0.8719 (0.6797)	-6.0686	11.3422	0.2001	2.0654	0.3177	2.0921	0.000 (0.000)	0.000 (0.000)
BcrqB	II	N(0,9)	1.1455 (0.8582)	0.4791	0.5613	0.0460	0.1976	-0.4702	0.5830	0.000 (0.000)	0.000 (0.000)
BcrqRB	II	N(0,9)	1.1545 (0.8548)	0.4910	0.5704	0.0446	0.1782	-0.4597	0.5830	0.000 (0.000)	0.000 (0.000)
crqL	III	N(0,9)	1.1852 (0.8794)	0.7163	0.7798	0.0745	0.2567	-0.7066	0.7914	0.000 (0.000)	0.0375 (0.0844)

BcrqL	III	N(0,9)	0.9141 (0.6852)	-0.9510	3.6893	-0.8278	2.6505	-0.0396	5.0624	0.000 (0.000)	0.0750 (0.1687)
BcrqB	III	N(0,9)	1.1747 (0.8715)	0.5402	0.5796	0.0928	0.2075	-0.4656	0.5125	0.000 (0.000)	0.0875 (0.1868)
BcrqRB	III	N(0,9)	1.1928 (0.8685)	0.5339	0.5715	0.0801	0.1795	-0.4570	0.5002	0.000 (0.000)	0.0875 (0.1868)
crqL	I	$t(3)$	1.1646 (0.8601)	0.2152	0.3639	-0.1129	0.2399	-0.1846	0.4292	0.000 (0.000)	0.0750 (0.1581)
BcrqL	I	$t(3)$	0.8979 (0.6797)	-0.9943	2.0870	-0.1859	1.7717	1.5153	2.6411	0.000 (0.000)	0.1500 (0.2342)
BcrqB	I	$t(3)$	1.1513 (0.8582)	0.0876	0.2830	-0.0043	0.1315	-0.0744	0.2436	0.000 (0.000)	0.1750 (0.2372)
BcrqRB	I	$t(3)$	1.1549 (0.8548)	0.0986	0.2876	0.0030	0.1367	-0.0853	0.2397	0.000 (0.000)	0.1500 (0.1936)
crqL	II	$t(3)$	1.1464 (0.8462)	0.3619	0.3967	0.0780	0.1985	-0.0856	0.3059	0.000 (0.000)	0.000 (0.000)
BcrqL	II	$t(3)$	0.8719 (0.6574)	-1.4362	2.0935	-0.7196	2.2226	0.1590	8.3805	0.000 (0.000)	0.000 (0.000)
BcrqB	II	$t(3)$	1.1262 (0.8306)	0.1042	0.1769	0.0767	0.1656	0.0358	0.1983	0.000 (0.000)	0.000 (0.000)
BcrqRB	II	$t(3)$	1.1244 (0.8251)	0.1017	0.1683	0.0771	0.1559	0.0442	0.1949	0.000 (0.000)	0.000 (0.000)
crqL	III	$t(3)$	1.1312 (0.8314)	0.1617	0.3041	-0.0052	0.1264	-0.2184	0.2754	0.000 (0.000)	0.0500 (0.1208)
BcrqL	III	$t(3)$	0.8656 (0.6523)	-5.6121	11.2557	0.0656	0.6537	1.1001	2.6790	0.000 (0.000)	0.0625 (0.1350)
BcrqB	III	$t(3)$	1.0921 (0.7944)	-0.0902	0.2396	-0.0336	0.1140	0.0141	0.1936	0.000 (0.000)	0.1000 (0.1419)
BcrqRB	III	$t(3)$	1.0866 (0.8047)	-0.0891	0.2423	-0.0391	0.1121	0.0183	0.1962	0.000 (0.000)	0.0875 (0.1449)
crqL	I	$\chi^2_{(3)}$	1.1312 (0.8314)	-2.1064	2.1273	-2.5903	2.6075	-3.4457	3.4717	0.000 (0.000)	0.1875 (0.1215)
BcrqL	I	$\chi^2_{(3)}$	0.8781 (0.6605)	-1.2729	4.7838	-5.5939	8.5012	-1.6062	3.7979	0.000 (0.000)	0.3375 (0.2503)
BcrqB	I	$\chi^2_{(3)}$	1.0921 (0.7964)	-2.3316	2.3573	-2.5182	2.5276	-3.1941	3.2031	0.000 (0.000)	0.3000 (0.2058)
BcrqRB	I	$\chi^2_{(3)}$	1.0866 (0.8082)	-2.3531	2.3798	-2.5156	2.5235	-3.1886	3.1983	0.000 (0.000)	0.2250 (0.1149)
crqL	II	$\chi^2_{(3)}$	1.1196 (0.8251)	-2.3412	2.3609	-2.7336	2.7530	-3.2627	3.2815	0.000 (0.000)	0.000 (0.000)
BcrqL	II	$\chi^2_{(3)}$	0.8719 (0.6574)	-1.8837	2.2778	0.2266	5.5702	-1.4045	4.7745	0.000 (0.000)	0.000 (0.000)
BcrqB	II	$\chi^2_{(3)}$	1.0725 (0.7954)	-2.5572	2.5774	-2.5538	2.5628	-2.9879	2.9970	0.000 (0.000)	0.000 (0.000)
BcrqRB	II	$\chi^2_{(3)}$	1.0769 (0.7974)	-2.5601	2.5827	-2.5507	2.5583	-2.9910	2.9993	0.000 (0.000)	0.000 (0.000)
crqL	III	$\chi^2_{(3)}$	1.1021 (0.8237)	-2.2466	2.2789	-2.7213	2.7467	-3.6372	3.666	0.000 (0.000)	0.0375 (0.1186)
BcrqL	III	$\chi^2_{(3)}$	0.8781 (0.6605)	-1.0208	4.0609	-4.2538	9.6318	-0.1436	2.9763	0.000 (0.000)	0.0500 (0.1208)
BcrqB	III	$\chi^2_{(3)}$	1.0725 (0.7954)	-2.5197	2.5346	-2.6753	2.6802	-3.2510	3.2692	0.000 (0.000)	0.0625 (0.1587)
BcrqRB	III	$\chi^2_{(3)}$	1.0769 (0.7974)	-2.5248	2.5408	-2.6610	2.6659	-3.2473	3.2641	0.000 (0.000)	0.0625 (0.1587)

Conclusion

In this paper, we present composite tobit QR with reciprocal bridge penalty from a Bayesian perspective. We introduce an efficient MCMC algorithm for Bayesian simultaneous covariate selection and estimation in tobit quantile regression based on a mixture representation of the inverse Laplace distribution for the regression coefficient. Specifically, the proposed approach uses prior distributions for the regression coefficients that are scale mixtures of inverse uniform priors on the coefficients and independent Gamma priors on their mixing parameters. The simulation results analysis show that the proposed method performs very well compared to the existing method.

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