How to Cite:

A Short Review of Decidability of Boolean Algebras And Structuree of Rational Numbers in Different Languages

Zahra Sheikhaleslami
Ph.D. Student of Tabriz University. The University of Tabriz P.O.Box 51666- 16471,Bahman 29thBoulevard; Tabriz; Iran.
zahra.sheikhaleslami@gmail.com

Saeed Salehi
Research Institute for Fundamental Sciences (RIFS),
University of Tabriz, 29 Bahman Boulevard,
P.O.Box 51666–17766, Tabriz, Iran.
School of Mathematics,
Institute for Research in Fundamental Sciences (IPM),
P.O.Box 19395–5746, Tehran, Iran.
root@saeedsalehi.ir
http://saeedsalehi.ir/

Abstract : This article consists of two parts.First,we study boolean algebras.
Boolean algebras are famous mathematical structures.Tarski showed the decidability of the elementary theory of Boolean algebras.In this paper, we study the different kinds of Boolean algebras and their properties. And we present for the first-order theory of atomic Boolean algebras a quantifier elimination algorithm. The subset relation is a partial order and indeed a lattice order,And I will prove that the theory of atomic Boolean lattice orders is decidable, and furthermore admits elimination of quantifiers.So the theory of the subset relation is

International Journal of Health Sciences ISSN 2550-6978 E-ISSN 2550-696X © 2022.
Corresponding author: *authorname; Email: email@gmail.com*
Manuscript submitted: … , Manuscript revised: … , Accepted for publication: …

decidable.And we will study decidability of atomlss boolean algebra.Second part,
Of this paper we show that the structure of rational numbers in different languages has the property of quantifier elimination,and hence is decidable.This proofes are organized in two parts. We first review some classical theorems and will give new proofs for old results.In second part,we will show decidability and axiomatization of the structure $\langle Q, \sqsubseteq \rangle$

Keywords: Boolean algebras, Decidability,Model Theory,Quantifier-Elimination

.

Introduction

A (non-strict) partial order is a binary relation $\leq$ *over a set* P *satisfying particular axioms which are discussed below. When* $a \leq b,$ we say that a is related to b .The axioms for a non-strict partial order state that the relation $\leq$is reflexive, antisymmetric, and transitive. That is, for all a, b, and c *in* P, it must satisfy:

- a $\leq$a (reflexivity: every element is related to itself).
- if a$\leq$b and b$\leq$a, then a=b (antisymmetry: two distinct elements cannot be related in both directions).
- if a$\leq$ b and b$\leq$c, then $a \leq c$ (transitivity: if a first element is related to a second element, and, in turn, that element is related to a third element, then the first element is related to the third element).

Boolean algebras was first intoduced by Boole in an effort to automate reasoning.Since that they have been extensively studied, and have proved fundemental in numerous application areas.We consider the structure $\langle B, \vee, \wedge, ', 0, 1 \rangle$where B is a set.$\vee, \wedge$ are binary operations in B, 'is a monary operation in B and$0,1 \in B$So, we have for $x, y, z \in B$:

$$x \vee y = y \vee x, \; x \wedge y = y \wedge x$$

$$x\vee (y\vee z)=(x\vee y)\vee z \quad x\wedge (y\wedge z)=(x\wedge y)\wedge z$$
$$x\vee x=x \quad x\wedge x=x$$
$$x=x \vee (x\wedge y) \quad x=x \wedge (x\vee y)$$
$$x\wedge (y\vee z)=(x\wedge y)\vee (x\wedge z)$$
$$x\vee (y\wedge z)=(x\vee y)\wedge (x\vee z)$$
$$x\wedge \quad = \quad x\vee \quad =$$
$$x\wedge x' = \quad x\vee x' =$$

An atom in Boolean algebra is an element x such that $x\neq 0$ and if y<x, then y=0. We will define a first order formula At(x) with meaning x is an atom:
$x\neq 0$ & $\neg$ y(y$\neq$0 \& x$\wedge$ y=y)
And if B be a Boolean algebra.It is said that B is atomic iff for every y $\in$ B,y$\neq$0, there is x$\in$B such that x is an atom and x$\leq$ y. I am trying to describe the concept of an atom in a Boolean algebra.let I={ a,b{be a set, and P(I)= {ϕ,{ a {,{ b {,{ a,b{{and A={ϕ,\{ a {,{ a,b{{be one of the possible algebras of subsets of I .we have A being an algebra of set, it is also a Boolean algebra.
} a {is atom.In any partially ordered set with a minimum element, an atom is element that covers the minimum element.And let X={ a,b,c{ be a set, and
A={ ϕ,{ a {,{ b {,X} *be one of the algebras of subsets of X*
Now, an element in is a atom if, for every y$\in$A , either x$\wedge$ y=x or x$\wedge$y=0. so } a{ and } b,c { are the atoms of A so, every singleton set is an atom.

Example 1.Let Z denote the set of integer,

$$B_Z=\langle \text{ Powerset } \{Z\}),\cup,\cap,', \phi,Z \rangle$$

is a Boolean algebra of sets.In this algebra for each} i{is an atom.This algebra is atomic.

We show that a finite Boolean algebra is made of its atoms.
Becase:

•Boolean algebras can be ordered by $x\leq y \Leftrightarrow x\vee y=y \Leftrightarrow x\wedge y=x$

•Atoms are exactly the minimal nonzero elements, i.e. a is an atom iff $0\leq a$ and
$0<x\leq a \Rightarrow x=a$.

•In a finite Boolean algebra each element is join of atoms such that below of it.

$$x=\vee \{ a\in B|\ a\leq x \ \&\ a \text{ is atom } \{$$

•By finiteness, if $z=x-y$, $z=x\wedge\neg y\neq 0$, so we have an atom below z.

Lemma1. A finite Boolean algebra is atomic.
proof:It is to be shown that every non-zero element p is atomic.If p itself is an atom, we are done. If not , then there must be a non-zero element p_1 strictly below p.If p_1
is an atom, then again we are done. If not ,there must be a non-zero element
p_2 strictly below p_1 ,and so on. Eventually this process must lead to an atom below p,otherwise, the Boolean Algebra would have an infinite , stiricly descending chain of elements, contradicting the assumption that the algebra is finit. Finite Boolean Algebras embedded into P(n).A Boolean Algebra is atomless if it has no atoms.Every atomless Boolean algebras with more than one element must be infinite.Indeed,the unit 1 is different from zero, so there is a non-zero element p_1 strictly below 1;otherwise, 1 would be an atom. Because $p_1$is not zero,there must be a non-zero element $p_2$strictly below p_1;otherwise,$p_1$decreasing sequence of elements $1> p_1> p_2>\dots$.
The interval algebra of the real numbers is atomless. Also the interval algebra of t\she rational numbers is atomless, or the regular open algebra of the sace of real numberes is atoless.
The axioms of the theory of atomless a Boolean algebrasare the universal quantification of the following formulas.

$x\wedge y=y\wedge x$, $x\vee y=y\vee x$
$x\wedge (y\wedge z)=(x\wedge y)$\land z, $x\vee (y\vee z)=(x\vee y)\vee z$
$(x\wedge y)\vee y=y$, $(x\vee y)\wedge y=y$
$x\wedge (y\vee z)=(x\wedge y)\vee (x\wedge z)$, $x\vee (y\wedge z)=(x\vee y\wedge (x\vee z)$
$x\wedge x^c=0$, $x\vee x^c=1$
$\neg At(x)$,$0\neq 1$
Recall that $At(x) \equiv x\neq 0$ & $\neg \exists$ $y(y\neq 0$ & $x\wedge y=y)$ and $0\neq 1$
We have mentioned the contents and theorems and examples from.[1,3,4]

Subsection 1: Deciding Boolean Algebras
At the study of Boolean algebras, we show decidability and undecidability questions for the class of Boolean algebras,And We describe an algorithm for deciding the Boolean algebras.A basic result of Tarski is that the elementary theory of Boolean algebras is decidable. Even the theory of Boolean algebras with a distinguished ideal is decidable. On the other hand, the theory of a Boolean algebra with a distinguished subalgebra is undecidable. Both the decidability results and undecidablity results extend in various ways to Boolean algebras in extensions of first-order logic.

Theorem1.Let P(I) is denot the power set of I. We have $(P(I),\subseteq,\cup,\cap,')$ is Boolean Algebra.
Because:
For $P,Q,R\subseteq I$,$P,Q,R\in P(I)$,we have:

$$P\cap Q=Q\cap P,\ P\cup Q=Q\cup P$$
$$P\cap (Q\cap R)=(P\cap Q)\cap R,\ P\cup (Q\cup R)=(P\cup Q\cup R$$
$$P\cap (Q\cup R)=(P\cap Q)\cup (P\cap R),\ P\cup (Q\cap R)=(P\cup Q)\cap (P\cup R)$$
$$P\cap I=P,\ P\cup \phi= P\$\backslash\backslash$$
$$P\cap P'=\phi,\ P\cup P'=I$$

The axioms of the theory of $(P(I), \subseteq, \cap,\cup,')$:
1) $a\subseteq a$
2) $a\subseteq b\subseteq a \rightarrow a=b$
3) $a\subseteq b \subseteq c \rightarrow a\subseteq c$
4) $z\subseteq x,y \leftrightarrow z\subseteq x\cap y$

5) x,y $\subseteq$z $\leftrightarrow$x$\cup$y $\subseteq$z
6) $\phi\subseteq$x
7) $\forall\, x\, \exists\, y\, (x \subset y)\;\; x \neq\, y$
8) b$\cap$ (a\ b)= ϕ
9) (a$\cap$b) $\cup$ (a\ b) =a$

Subsection 2:An algorithm for deciding the theory Boolean algebras.

We present an algorithm and show how decide.We have some definitions:

•L={$\subseteq$, $\cap$, $\cup$, A\ B,=, ϕ , C_n,E_n, $\in$ N^+}
• A (a) $\leftrightarrow \forall\, x[x \subseteq\, a \;\rightarrow\; x = \phi \vee\, x = a] \wedge\, a \neq \phi$
•$C_n(x) \equiv\, \exists\, a_1 ... a_n\, \left(\wedge_{i<j} a_i \neq\, a_j \wedge\, \wedge^n{}_{i=1} A\, a_i\, \wedge\, \wedge_{i=1}^n a_i \subseteq\, X\right)$
• $E_n(x)\;\equiv C_n(x) \wedge\, \neg\, C_{n+1}(x)$
•The next step of the algorithm is eliminate =:
Because: a=b $\Leftrightarrow$ a$\subseteq$b $\wedge$b$\subseteq$a
•eliminate $\subseteq$
Because: a$\subseteq$b$\Leftrightarrow$a\ b=$\phi \leftrightarrow\, E_0(\, a - b)$
•And eliminat:$\neg$
Because:
$\neg$Cn (x) $\leftrightarrow\;\; \wedge_i E_i(x)$
$\neg$ E n (x) $\leftrightarrow\; C_{n+1}(x) \vee\;\; \wedge_i E_i(x)$
Quantifier-Elimination for Boolean formulas is as follows:
• L$=\{\; \cap\;\; \cup\;\;\;\; =\;\;\; \{C_n\}\; \{E_n\}\; n \in\, N^+\}$
We have the following\\
•R$=\{=|\{C_n\}_{n\geq 0}| \subseteq | \{\, E_n\}_{n\geq 0}\}$
•F$=\{A|F_1 \wedge\, F_2\; |F_1 \vee\, F_2| \neg\, F| \exists\, F\, | \forall\, F\}$
•A$=\{B_1 = B_2 |B_1 \subseteq\, B_2 |C_n(B), E_n(B)\}$
•B $\begin{matrix} = \{x|\phi|I|B_1 \cap\, B_2|B_1 \cup\, B_2\, |\, B^c\} \\ n = \{\;\; ||\;\; |\;\;\;\; \} \end{matrix}$

So it is enough to consider only the following formulas:
$C_n(x)=|x|\geq n$ $E_n(x)=|x|=n$ Contradictions of liters are eliminated according to the above definitions.
$\neg|x|= n \leftrightarrow |x|= 0\vee ... \vee |x|= n-1 \vee |x|\geq n+1$
$\neg |x|\leq n \leftrightarrow |x|= 0\vee ...\vee |x|= n-1$
So at this step we've removed some of the relationships as follow:

1.Eliminate equality: $a=b\leftrightarrow a\subseteq b \wedge b\subseteq a$

2. Delete inclusion: $a\subseteq b \leftrightarrow |a\cap b^c|=$

3.Eliminate contradictions:
$\neg C_n(x)\leftrightarrow \vee_{i<n}E_i(x)$
$\neg E_n(x) \leftrightarrow C_{n+1}(x)\vee \vee_{i<n}E_i(x)$

Language to Quantifier-Elimination:

$$\cap \cup^c \phi \{C_n\}_{n\geq 0} \{E_n\}_{n\geq 0}$$

term:

$$x \phi \cap \cup^c$$

Quantifier Elimination:
In theresulting formula, each set variable x occurs in some term |t(x)|. each set expression |t(x)| as a union ofcubes (regions in the Venn diagram). The cubes have the form $\cap_{i=1}^n \ {}_i^{\alpha_i}$ where $x_i^{\alpha_i}$ is either x_i or x_i^c ; there are m=2^n cubes. The resulting formula is then equivalent to:

$$\exists x(\wedge_i C_{n_i}(t_i(x))\wedge\left(\vee_j E_{n_j}(t'_j(x))\right)$$

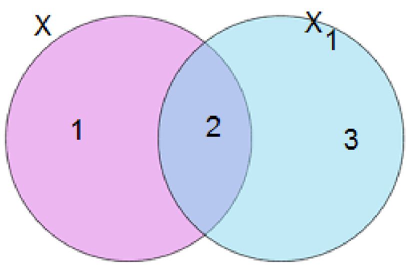
for example:

$$\exists x\left(|x \cap c| \leq \quad \wedge \ |x \cap c| \leq \quad \wedge \ |c - x| = \quad \right)$$
$$\exists x\left(C_3(x \cap c) \wedge C_7(x \cap c) \ \wedge E_2(c - x)\right) \equiv C_9(c)$$
$$\exists x\left(C_5(x \cap c) \wedge C_7(x \cap d) \ \wedge E_6(c - x)\right) \equiv C_{11}(c) \wedge C_7(d)$$

More explained in the table below:
TABLE I

The main formula	Deleted form
$\exists y \ldots |x \cap y| \geq k \wedge |x \cap y^c| \geq l \ldots$	$|x| \geq k + l$
$\exists y \ldots |x \cap y| = k \wedge |x \cap y^c| \geq l \ldots$	$|x| \geq k + l$
$\exists y \ldots |x \cap y| \geq k \wedge |x \cap y^c| = l \ldots$	$|x| \geq k + l$
$\exists y \ldots |x \cap y| = k \wedge |x \cap y^c| = l \ldots$	$|x| = k + l$

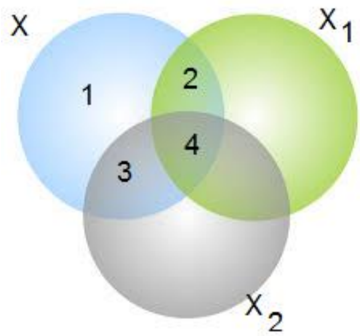
Subsection 2: Examples
We have shown with the following examples:
Example.1.We have sets X, X_1. (Shown in Figure 1)



(Figure 1)

$$Cn_1(\ )\wedge C_{n_2}(\ )\ \wedge C_{n_3} \equiv C_{n_2+n_3}(X_1)$$
$$Cn_1(\ )\wedge C_{n_2}(\ )\ \wedge E_{n_3} \equiv C_{n_2+n_3}(X_1)$$
$$Cn_1(\ )\wedge E_{n_2}(\ )\ \wedge C_{n_3} \equiv C_{n_2+n_3}(X_1)$$
$$Cn_1(\ )\wedge E_{n_2}(\ )\ \wedge E_{n_3} \equiv E_{n_2+n_3}(X_1)$$
$$En_1(\ )\wedge C_{n_2}(\ )\ \wedge C_{n_3} \equiv C_{n_2+n_3}(X_1)$$
$$En_1(\ )\wedge C_{n_2}(\ )\ \wedge E_{n_3} \equiv C_{n_2+n_3}(X_1)$$
$$En_1(\ )\wedge E_{n_2}(\ )\ \wedge C_{n_3} \equiv C_{n_2+n_3}(X_1)$$

Example.2.We have sets X,X_1, X_2. (Shown in Figure 2)



(Figure 2)

$$C_{n_1}(\ )\wedge C_{n_2}(\ )\ \wedge C_{n_3}(\ )\ \wedge C_{n_4}(\ )\equiv C_{n_2+n_4}(X_1)\wedge C_{n_3+n_4}(X_2)$$
$$C_{n_1}(\ )\wedge C_{n_2}(\ )\ \wedge C_{n_3}(\ )\ \wedge E_{n_4}(\ )\equiv C_{n_2+n_4}(X_1)\wedge C_{n_3+n_4}(X_2)$$
$$C_{n_1}(\ )\wedge C_{n_2}(\ )\ \wedge E_{n_3}(\ )\ \wedge C_{n_4}(\ )\equiv C_{n_2+n_4}(X_1)\wedge C_{n_3+n_4}(X_2)$$
$$C_{n_1}(\ )\wedge C_{n_2}(\ )\ \wedge E_{n_3}(\ )\ \wedge E_{n_4}(\ )\equiv C_{n_2+n_4}(X_1)\wedge C_{n_3+n_4}(X_2)$$
$$C_{n_1}(\ )\wedge E_{n_2}(\ )\ \wedge C_{n_3}(\ )\ \wedge C_{n_4}(\ )\equiv C_{n_2+n_4}(X_1)\wedge C_{n_3+n_4}(X_2)$$

$$C_{n_1}(\ )\wedge E_{n_2}(\ )\ \wedge E_{n_3}(\ )\ \wedge C_{n_4}(\ )\equiv C_{n_2+n_4}(X_1)\wedge C_{n_3+n_4}(X_2)$$
$$C_{n_1}(\ )\wedge E_{n_2}(\ )\ \wedge E_{n_3}(\ )\ \wedge E_{n_4}(\ )\equiv C_{n_2+n_4}(X_1)\wedge C_{n_3+n_4}(X_2)$$
$$E_{n_1}(\ )\wedge C_{n_2}(\ )\ \wedge C_{n_3}(\ )\ \wedge C_{n_4}(\ )\equiv C_{n_2+n_4}(X_1)\wedge C_{n_3+n_4}(X_2)$$
$$E_{n_1}(\ )\wedge C_{n_2}(\ )\ \wedge E_{n_3}(\ )\ \wedge C_{n_4}(\ )\equiv C_{n_2+n_4}(X_1)\wedge C_{n_3+n_4}(X_2)$$
$$E_{n_1}(\ )\wedge C_{n_2}(\ )\ \wedge C_{n_3}(\ )\ \wedge E_{n_4}(\ )\equiv C_{n_2+n_4}(X_1)\wedge C_{n_3+n_4}(X_2)$$
$$E_{n_1}(\ )\wedge E_{n_2}(\ )\ \wedge C_{n_3}(\ )\ \wedge C_{n_4}(\ )\equiv C_{n_2+n_4}(X_1)\wedge C_{n_3+n_4}(X_2)$$
$$E_{n_1}(\ )\wedge E_{n_2}(\ )\ \wedge E_{n_3}(\ )\ \wedge C_{n_4}(\ )\equiv C_{n_2+n_4}(X_1)\wedge C_{n_3+n_4}(X_2)$$
$$E_{n_1}(\ )\wedge E_{n_2}(\ )\ \wedge C_{n_3}(\ )\ \wedge E_{n_4}(\ )\equiv C_{n_2+n_4}(X_1)\wedge C_{n_3+n_4}(X_2)$$
$$E_{n_1}(\ )\wedge E_{n_2}(\ )\ \wedge E_{n_3}(\ )\ \wedge E_{n_4}(\ )\equiv C_{n_2+n_4}(X_1)\wedge C_{n_3+n_4}(X_2)$$

Subsection 3: atomless boolean algebra

we have the interval algebra of rational number .the intrval algebra of the rational number is aomless.

lemma1.We have in every Boolean algebra:

$$p\subset\ P\ \ q\subset Q\ \ P\cap\ Q=\phi\ \ P\ \ Q=\quad\rightarrow\ p+q\subset P+Q$$

Proof:

p+q$\subseteq$ P+Q .To show: p+q$\neq$\ P+Q We assume p+q= P+Q so, (P+Q).

$\overline{p}=P.\ \overline{p}+Q.\overline{p}$,Because $Q\cap\ p=\phi$ we have:

$=P.\overline{p}+Q=(p+q).\ \overline{p}$

$p\ \ \overline{p}+q\ \ \overline{p}$

$=q(q\cap\ p\ \ =\phi)$

q=

$(\ P+Q)\ \ \overline{p}\,\overline{q}\ \ =q\,\overline{q}=\phi$

$P\ \overline{p}\,\overline{q}+Q\ \overline{p}\,\overline{q}=\quad\Leftrightarrow\ P\,\overline{q}+Q\ \overline{p}=$

$\Leftrightarrow\ P\,\overline{q}=\ Q\,\overline{p}=$

Which contradicts with the assumption

lemma2. The following formulas are equivalent:

$$\exists x\left(rx = \quad \wedge\ s\bar{x} = \quad \wedge\ \wedge_{i=1}^{m} u_i x \neq \quad \wedge\ \wedge_{j=1}^{n} v_j \bar{x} \neq \quad \right)$$

$$\equiv\ rs = 0 \backslash land\ \ \backslash exists\ y\left(\ \wedge_{i=1}^{m} u_i \bar{r} y \neq\ 0\ \wedge\ \wedge_{j=1}^{n} v_j \bar{s}\bar{y} \neq\ 0\right)$$

Proof:

$\Rightarrow$

If there is x such that,

$rx = 0 \wedge\ s\,\bar{x} = \quad \wedge\ \wedge_{i=1}^{m} u_i x \neq \quad \wedge\ \wedge_{j=1}^{n} v_j \bar{x} \neq \quad)$

$rx = \quad \wedge\ rs\bar{x} = \quad \rightarrow\ rs\left(x + \bar{x}\right) = \quad \rightarrow rs(\ \ ) = \quad \rightarrow\ rs =$

$u_i x \neq \quad \rightarrow u_i x\left(r + \bar{r}\right) \neq$

$\rightarrow u_i x r + u_i x\ \bar{r}\ \neq\ 0$

$u_i x\ \bar{r}\ \neq\ 0$

$\Rightarrow \exists\ x\left(\ \wedge_{i=1}^{m} u_i \bar{r} x \neq \quad \wedge\ \wedge_{j=1}^{n} v_j \bar{s}\bar{r} \neq \quad \right)$

$\Leftarrow$

Suppose rs=0 , there is y such that ,

$\wedge_{i=1}^{m} u_i \bar{r} y \neq \quad \wedge\ \wedge_{j=1}^{n} v_j \bar{s}\bar{y} \neq$

We put,

$$x = \bar{r}\ \ (s + y)$$

$$\bar{x}\ = r + \bar{s}\,\bar{y} = \left(r + \bar{s}\right)\left(r + \bar{y}\right)$$

$$\bar{s}\ \left(r + \bar{y}\right)$$

We show,

$\wedge_{i=1}^{m} u_i x \neq \quad \wedge\ \wedge_{j=1}^{n} v_j \overline{x \neq}$

$u_i x =\ u_i\,\bar{r}\left(s + y\right) = u_i\,\bar{r}s + u_i\,\bar{r}y \supseteq\ u_i\,\bar{r}y \neq$

$v_j\,\bar{x} = v_j\,\bar{s}\left(r + \bar{y}\right) = v_j\,\bar{s}r + v_j\,\bar{s}\,\bar{y} \supseteq v_j\,\bar{s}\,\bar{y}\ \neq$

Theorem1.The theory of atomless Boolean algebra in the language $L=\langle \quad \wedge \vee \neg = \rangle$ accepts the quantifier elimination.

:Proof

$F=\{A|F_1\wedge F_2 \ |F_1\vee F_2|\neg F|\exists F \ | \forall F\}$

$A=\{t_1=t_2\}$

$T=\{x|0|1|t_1\vee t_2|t_1\wedge t_2 \ |\neg t\}$

we have: t=s

and t$\neq$ s so

$$t=\bigcup_{i\in I}\left(\bigcap_{j\in J} a_{\{i,j\}}\right)$$

such that $a_{i,j}$ Is variable or complement variable.

Terms included x: $x \ r+\bar{x} \ s$

Atomic formulas:

$t=s \Leftrightarrow t \ \bar{s}+\bar{t} \ s=$

$t\neq s \Leftrightarrow t \ \bar{s}+\bar{t} \quad \neq$

Atomic formulas include x :

r. x+s.$\bar{x}$=0

$\Leftrightarrow$ r. x=0 $\wedge$ s.$\bar{x}$=0

$\Leftrightarrow x\subseteq\bar{r}\wedge s\subseteq x\leftrightarrow sr=0\wedge s\subseteq x\Leftrightarrow sr=\phi$

Contradiction of atomic formulas include x :

$r \ x+s\bar{x}\neq \quad \Leftrightarrow r \ x\neq \quad \vee s\bar{x}\neq$

so it is enough to eliminate quantifiers of the folloeing formulas:

$$\exists x\left(rx= \quad \wedge s\bar{x}= \quad \wedge \wedge_{i=1}^{m} u_i x\neq \quad \wedge\wedge_{j=1}^{n} v_j\bar{x}\neq \quad\right)$$

$$\equiv rs= \quad \wedge \exists y\left(\wedge_{i=1}^{m} u_i\bar{r}y\neq \quad \wedge \wedge_{j=1}^{n} v_j\bar{s}\bar{y}\neq \quad\right)$$

becuase:

$\Rightarrow$

if there is x such that

rs=0 $\wedge\ \exists y\left(\ \wedge_{i=1}^{m} u_i \bar{r} y \neq\ 0\ \wedge\ \wedge_{j=1}^{n} v_j \bar{s}\bar{y} \neq\ 0\right)$ so rsx=0 , rs$\bar{x}$ =0$\Rightarrow$ rs(x+ $\bar{x}$)=0$\Rightarrow$rs=0

$u_i x \neq\ 0 \rightarrow\ u_i x\left(r+\bar{r}\right)\ \neq\ 0$

$\Rightarrow u_i xr + u_i x\bar{r} \neq$

$\Rightarrow\ u_i x\bar{r} \neq$

$v_j \bar{x}\ \neq\quad \rightarrow\ v_j \bar{x}\left(s+\bar{s}\right) \neq$

$\Rightarrow\ v_j \bar{x}s + v_j \bar{x}\bar{s}\ \neq$

$\Rightarrow\ v_j \bar{x}\bar{s}\ \neq$

$\Rightarrow\ \exists x\left(\ \wedge_{i=1}^{m} u_i \bar{r} x \neq\quad \wedge\ \wedge_{j=1}^{n} v_j \bar{x}\bar{s}\ \neq\quad\right)$

$\Leftarrow$:

we assume there was $rs \neq$ and y such that

$\wedge_{i=1}^{m} u_i \bar{r} y \neq\ 0\ \wedge\ \wedge_{j=1}^{n} v_j \bar{s}\bar{x} \neq\ 0$

we put

$x = \bar{r}\left(s+y\right)$

$\bar{x} = r + \bar{s}\bar{y} = \left(r+\bar{s}\right)\ \left(r+\bar{y}\right)$

$= \bar{s}\quad \left(r+\bar{y}\right)$

so

rx=0 ,s$\bar{x}$=0

it is enough to show

$\wedge_{i=1}^{m} u_i x \neq\quad \wedge\ \wedge_{j=1}^{n} v_j \bar{x} \neq$

$u_i x =\ u_i \bar{r}\left(s+\bar{y}\right) =\ u_i \bar{r}s +\ u_i \bar{r}y\ \supseteq\ u_i \bar{r}y$

$v_j \bar{x}\ =\ v_j \bar{s}\left(r+\bar{y}\right) = v_j \bar{s}r + v_j \bar{s}\bar{y} \supseteq\ v_j \bar{s}\bar{y} \neq$

so it suffices to eliminate the quantifier of the formula

$\exists y\left(\ \wedge_{i=1}^{m} a_i y \neq\quad \wedge\ \wedge_{j=1}^{n} b_j \bar{y} \neq\quad\right)$

$\equiv\ \ \wedge_{i=1}^{m} a_i \neq\quad \wedge\ \wedge_{j=1}^{n} b_j \neq$

$\Rightarrow$

it is obviously.

$\Leftarrow$

we consider all cells C_α including a_i, $b_j$both cells are distinctly distinct

$C_\alpha \cap C_\beta = \phi$ C_α $C_\beta =$

each set is equall tocommunity of cells contained in it

$Z = \sum_{C_\alpha \subseteq Z} C_\alpha$ for all Zand any cell C we have C$\subseteq$ Z.

with

$C \subseteq \overline{Z}$,

$Z \neq \leftrightarrow \exists\, \alpha \left(C_\alpha \subseteq Z \wedge C_\alpha \neq \right)$

for any cells is not equall zero Cα from being atomless dα there is such that

$\neq d_\alpha \subseteq C_\alpha \neq$ if Cα =0 put

dα =0,we put it now $y = \sum_{C_\alpha \neq 0} d_\alpha$

$a_i y \neq 0 : a_i y = a_i \sum_{C_\alpha \neq 0} d_\alpha = \sum_{C_\alpha \neq 0} a_i d_\alpha$

$\supseteq a_i d_\beta \supseteq c_\beta d_\beta = d_\beta \neq$

$\neq a_i = \sum_{C_\beta a_i} C_\beta$

$\exists\, \beta \ C_\beta \subseteq a_i \wedge C_\beta \neq$

$\neq d_\beta \subseteq C_\beta \neq$

$b_j \overline{y} \neq \quad b_j \overline{y} = b_j \prod_{C_{\alpha \neq 0}} \overline{d_\alpha} = \prod_{0 \neq C_\alpha} b_j \overline{d_\alpha}$

$\neq d_\beta \subset C_\beta \neq$

$b_j \overline{y} \neq \quad b_j \overline{y} = b_j \prod_{C_\alpha \neq 0} \overline{d_\alpha} = b_j \overline{d_\alpha}$

Forall C$_\alpha$ we have $C_\alpha \subseteq b_j$ with $C_\alpha \subseteq \overline{b_j} \Rightarrow \overline{C_\alpha} \supseteq b_j$ if $C_\alpha \neq$ then

$d_\alpha \subseteq C_\alpha \Rightarrow \overline{d_\alpha} \supseteq \overline{C_\alpha} \supseteq b_j$ so $b_j \overline{d_\alpha} = b_j$

$b_j \overline{y} = \prod_{C_\alpha \subseteq b_j} {}_j\overline{d_\alpha} = \prod_{0 \neq C_\alpha \subseteq b_j} {}_j\overline{d_\alpha}$

C$_\alpha$ =0$\Rightarrow d_\alpha = 1$

$C_\alpha \neq \quad \Rightarrow \quad \neq d_\alpha \subseteq C_\alpha \subseteq b_j$

$\sum_{0 \neq C_\alpha \subseteq b_j} d_\alpha \subset \sum_{0 \neq C_\alpha \subseteq b_j} C_\alpha = b_j$

$b_j \overline{\sum d_\alpha} \neq$
$b_j \prod_{0 \neq C_\alpha \subseteq b_j} d_\alpha \neq$
$b_j \bar{y} \neq$

The above proof we proved theory of Boolean algebras by the quantifier-elimination is decidable.

Subsection 4: Structure of Rational Numbers in Different Languages

A rational number is a number that can be in the form $\frac{p}{q}$ where p and q are integers and q is not equal to zero. All fractions, both positive and negative, are rational numbers.

A few examples are45,−78,134,and−203Each numerator and each denominator is an integer.Are integers rational numbers? To decide if an integer is a rational number, we try to write it as a ratio of two integers. An easy way to do this is to write it as a fraction with denominator one $\frac{3}{1}$,-8=$\frac{-8}{1}$.Since any integer can be written as the ratio of two integers, all integers are rational numbers.

Theorem1.Theory (Q,<) admits elimination of quantifier..
Proof:
Step 1: Identify the terms:
In structure)Q;<) every term involving x is equal to,

$$n. x+t \ (n \in N)$$

where x does not appear in t.
Step 2: Identify Atomic Formulas and Delete $\neg$ if possible:
All atomic formulas are,

u< v
u = v

First, we eliminate the inequality behind the atoms. Because,

$x \neq y \Leftrightarrow x < y \wedge y < x$

$x \quad not < y \Leftrightarrow x = y \vee y < x$ \end{flushleft}

Step 3: Simplify atomic formulas:
So the following formula must be eliminated quantifier.

$$\exists x\left(\wedge_i r_i < m_i \ x + s_i \wedge \ \vee_j n_j \ x + t_j < u_j \wedge \ \vee_l k_l \ x + v_l = wl\right)$$

Step 4:Uniform the coefficients x :

Let M is Multiply the coefficients by x .

$$M = \prod_i m_i \prod_j n_j \prod_l k_l$$

$$r_i \frac{M}{m_i} < M \ x + \frac{M}{m_i} s_i$$

$$M \ x + \frac{M}{n_j} t_j < \frac{M}{n_j} u_j$$

$$Mx + \frac{M}{k_l} v_l = \frac{M}{k_l} w_l$$

So the following formula admits quantifier elimination:

$$\exists x\left(\wedge_i r'_i < Mx + s'_i \wedge \ \wedge_j \ Mx + t'_j < u'_j \wedge \ \wedge_l \ Mx + v'_l = w'_l\right)$$

Step 5: Remove the coefficient x :
We put y=Mx .So, we have

$$\exists y\left(\wedge_i r'_i < y + s'_i \wedge \ \wedge_j \ y + t'_j < u'_j \wedge \ \wedge_l \ y + v'_l = w'_l\right)$$

We use the following equations

$$t = s \;\Leftrightarrow\; ct = cs$$
$$t < s \;\Leftrightarrow\; ct < cs$$

So

$$\exists\, x\left(\wedge_i r_i < \; x + s_i \wedge \;\; \wedge_j \;\; x + t_j < \; u_j \wedge \;\; \wedge_l \; x + v_l = w_l\right)$$

Step 6: Identification Phrases included x :

$$r_i < x + s_i \;\Leftrightarrow\; r_i + t_j + v_l < \; x + s_i + t_j + v_l$$
$$x + t_j < \; u_j \;\Leftrightarrow x + s_i + t_j + v_l < \; u_j + s_i + v_l$$
$$x + v_l = w_l \Leftrightarrow \; x + s_i + t_j + v_l = s_i + t_j + w_l$$
$$P = \; s_i + t_j + v_l$$

So

$$\exists\, x\left(\wedge_i r'_i < \; x + P \wedge \;\; \wedge_j \;\; x + P < \; u'_j \wedge \;\; \wedge_l \; x + P = w'_l \right)$$

we put y=x+P so we have,

$$\exists\, y\left(\wedge_i r'_i < \; y \wedge \;\; \wedge_j \;\; y < u'_j \wedge \wedge_l y = w'_l\right)$$

Therefore, it is enough to delete the quantifier in the following formula:

$$\exists\, x\left(\wedge_i r_i < \; x \wedge \;\; \wedge_j \;\; x < \; u_j \wedge \;\; \wedge_l \; x = w_l\right)$$

Step 7: Identify the states:

$$l \neq \quad \equiv \wedge_i r_i < w_0 \wedge \wedge_j w_0 < u_j \wedge \wedge_l w_0 = w_l \equiv True$$

$$l = \quad \equiv \exists x\left(\wedge_i r_i < x \wedge \wedge_j x < u_j\right)$$

$$l = j = \quad \equiv \exists x\left(\wedge_i r_i < x\right) \equiv True$$

$$l = i = \quad \equiv \exists x\left(\wedge_j x < u_j\right) \equiv True$$

$$l = \quad i \; j \neq \quad \equiv \exists x\left(\wedge_i r_i < x \wedge \wedge_j x < u_j\right) \equiv \wedge_i \wedge_j r_i < u_j \equiv True$$

Theorem2:The Theory of Addition)Q,+) admits elimination of quantifier.
Proof:
Step 1: Identify the terms:In structure)Q,+) every term involving x is equal to

$$n. x+t , (n \in N)$$

Where x does not appear in t .
Step 2: Identify Atomic Formulas:
All atomic formulas are,

$$u < v$$
$$u \neq = v$$

Step 3: Simplify atomic formulas:
So the following formula must be eliminated quantifier.

$$\exists x\left(\wedge_i k_i \; x + v_i = w_i \wedge \wedge_j m_j \; x + n_j \neq s_j\right)$$
$$\equiv \exists x\left(\wedge_i k_i \; x = u_i \wedge \wedge_j m_j \; x \neq t_j\right)$$

Step 4:Uniform the coefficients x .
Let M is Multiply the coefficients by x .

$$M = \prod_i k_i \; m_j$$

So the following formula admits quantifier elimination:

$$\exists\, x(\wedge_i M \ \ x = u_i^{'} \wedge \ \wedge_j M \ x \neq t_j^{'}$$

Step 5: Remove the coefficient x :
Put y=Mx .So, we have

$$\exists\, y(\wedge_i y = u_i^{'} \wedge \ \wedge_j y \neq t_j^{'})$$

We use the following equations

$$t = s \iff ct = cs$$
$$t \neq s \iff ct \neq cs$$

So

$$\exists\, x\Big(\wedge_i x = u_i \wedge \ \wedge_j x \neq \ t_j \Big)$$

Step 6: Identify the states:

$$i \neq \quad \equiv \ \wedge_i u_0 = u_i \wedge \ \wedge_j u_0 \neq \ t_j \ \equiv True$$
$$i = \quad j \neq \quad \equiv \ True$$

Theorem3.the Theory $\langle \ \ \mathbb{Q} \ + - \ \ < \ \rangle$ admites quantifier - elimination. and so has decidable theory .
Proof:
The following formula must be eliminated quantifier.

$$\exists x\Big(\wedge_i n_i \ \ x = t_i \wedge \ \wedge_j \ \ < m_j \ \ x + s_j \Big)$$

Similar to previous proofs, admites quantifier - elimination. and so has decidable theory .

Theorem4. the Theory $\langle \; \mathbb{Q}^{+} \;\; \times \qquad {}^{-1} \;\; R_{n} \;\; {}_{n\geq 2} < \; \rangle$ admites quantifier - elimination. and so has decidable theory .
Proof: [10]
Similar to previous proofs, admites quantifier - elimination. and so has decidable theory .

Theorem5:The theory of the rational numbers ($\mathbf{Q}$;$\trianglelefteq$) is decidable, and moreover axiomatizable.

Proof: quantifier elimination for The theory of the rational numbers ($\mathbf{Q}^{+}$; $\trianglelefteq$):

$$p \sqsubseteq q \;\; \Leftrightarrow \; \exists \, m \in \; \mathbb{N}^{+} \left(p \;\; m = q \right)$$

Structure ($\mathbf{Q}^{+}$;$\trianglelefteq$) is equivalent With structure $\left(\mathbb{Q}^{+} \;\; \times \right)$ First,We conclude decidablity $\left(\mathbb{Q}^{+} \;\; \times \right)$ of paper [10] .
so,the structure($\mathbf{Q}^{+}$;$\trianglelefteq$) Based on the article [10] is decidable .

We will express the axioms of rational numbers as follows:
Positive rational numbers are formed from two parts, the integer part whose denominator is one, and the Intrevel Algebra of rational numbers. The positive part of all the properties of natural numbers.So we have the axioms of ($\mathbf{N}$;$\trianglelefteq$) and atomless Boolean Algebra and the axioms of $\left(\mathbb{Q}^{+} \;\; \times \right)$ so we have the following axioms for($\mathbf{Q}^{+}$;$\trianglelefteq$):

[1] $\forall$ x $(x \sqsubseteq x)$
[2] $\forall$ x,y $(x \sqsubseteq y \sqsubseteq x \rightarrow x = y$
]3] $\forall$ x,y,z $(x \sqsubseteq y \sqsubseteq z \rightarrow x \sqsubseteq z)$
]4] $\forall$ x,y $\exists z \;\; (z \sqsubseteq x; y \;\wedge\; \forall \, t \;\; [t \sqsubseteq x; y \rightarrow t \sqsubseteq z]); \; z = x \sqcap y$
]5] $\forall$ x,y $\exists z \;\; (x; y \sqsubseteq z \;\wedge\; \forall \, t \, [x; y \sqsubseteq t \;\rightarrow \backslash z \sqsubseteq t \;)$, $z = x \sqcup y$
]6] $\forall$ x $(1 \sqsubseteq x)$
]7] $\forall$ x,y [$\forall$ z(SI(z)[z $\sqsubseteq x \rightarrow$ z $\sqsubseteq$ y]) $\rightarrow$ x $\sqsubseteq$ y]

]8] $\forall$ x ,y,z (SI* (x) $\wedge$ SI* (y) $\wedge$ SI* (z) $\wedge$ [(x,y $\sqsubseteq$ z $\vee$ $z \sqsubseteq x, y \rightarrow x \sqsubseteq y \vee y \sqsubseteq x$)
]9] $\forall$ x ,a ([SI* (a) $\wedge a \sqsubseteq x$] $\rightarrow \exists b$ (SI* (b) $\wedge a \sqsubseteq b \sqsubseteq x \wedge \forall$ c(SI* (c) $\wedge \sqsubseteq x; a$) $\rightarrow c \sqsubseteq b$])
]10]VAL(x,a $\wedge$ VAL(y,b) $\wedge$ [(a=b=1) $\vee$ (a=1 $\wedge$ b $\neq$ 1 $\wedge \forall$ x [SI* (c) $\wedge$ b $\sim$ c] $\rightarrow$ c $\not\sqsubseteq$ x) $\vee$ ((1 $\sqsubseteq a \sqsubseteq b$)] $\rightarrow VAL(x \sqcap y; a)\&VAL(x \sqcup y; a)$.
]11] $\forall$ x (x $\neq$ 0 $\rightarrow \exists$ a (P(a) $\&a \sqsubseteq x$))
]12] $\forall$ x (x $\neq$ 0 $\rightarrow \exists$ a(P(a) \& $\&a \not\sqsubseteq x$)
]13] $\forall$ x $\exists$ s $\forall$ a (P(a) $\rightarrow$ (V(a,x) $\neq$ 0 $\rightarrow$ V(a,s) $\neq$ a)\& (V(a,x)=0 $\rightarrow$ V(a,s)=0)), s= SUPP (x) .
]14] $\forall$ x, y $\exists$ z \forall a (P(a) $\rightarrow$ ((a $\not\sqsubseteq$ x $\rightarrow$ V(a,z)= V(a,y)) & a $\sqsubseteq$ x $\rightarrow$ V(a,z)= 0))), z = $\overline{T}$ (x,y) .
] 15-1] $\forall$ a, x(SI(a,x) $\rightarrow \exists$ y (SI(a,y)\& x $\sqsubseteq y \& y \neq$ x & $\forall$ z ((SI(a,z)\& z)&x $\sqsubseteq$ z) $\rightarrow$ y $\sqsubseteq$ z))), y = S_a(x) .
]15-2] $\forall$ a, x(SI(a,x) $\wedge$ x $\neq$ 0) $\rightarrow \exists$ y (SI(a,y)\& S_a (y)=x)), y = P_a(x)
] 16] $\forall$ x $\exists$ y $\forall$ a (P_a $\rightarrow$ ((a $\not\sqsubseteq$ x $\rightarrow$ V(a,y)=0)\& ((a $\sqsubseteq$ x $\rightarrow$ V(a,y)= S_a V(a,x)))) , y= I (x)
]17] $\forall$ x $\forall$ y $\exists$ z $\forall$ a (P(a) $\rightarrow$ (V(a,z)=0 or a \& V(a,z)=a $\rightarrow$ ((a $\sqsubseteq$ x or $\sqsubseteq$ y)& V(a,x) $\sqsubseteq$ V(a,y)))
[18]$x \sqcap y = y \sqcap x, x \sqcup y = y \sqcup x$
[19]$x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$ $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
]20]($x \sqcap y) \sqcup y = y$,$(x \sqcup y) \sqcap y = y$
]21] $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$, $x \sqcup (y) = (x \sqcup y) \sqcap (x \sqcup z)$
[22]x $\sqcap x^{-1} = 1$, $x \sqcup x^{-1} = x$
[23] $\neg P(a)$
[24] $\forall$ x,y,z(x.(y.z)=(x.y).z)
[٢٥] $\forall$ x(x.1=x)
[26] $\forall$ x(x.x^{-1}=1)
[27] $\forall$ x,y(x.y=y.x)
[28] $\forall$ x(x^n=1 $\rightarrow x=1$)
]29] $\forall$ x,y(0<x<y $\rightarrow \exists$ z(x^{2n}<z<$y^{٢n}$)),n $\geq$ 1 $
]30] $\forall$ v١,...,v,1 $\exists$ x $\forall$ z $\wedge_{k=1}^{l}$ x^n. $v_k \neq$ $z^m{}_k$

Conclusion

Boolean algebras are mathematical structures important in many branches of mathematics and computer science.Boolean algebras are closely related to Boolean lattices and Boolean rings.We prove here that the theory of atomic Boolean algebras is decidable and furthermore admits elimination of quantifiers down to the language including the Boolean operations and the relations expressing the height or size of an object, $|x| \geq n$ and $|x| = n$ .The structure of $(Q,<)$ is decidable ,by quantifier elimination .The theory (Q,+) admites quntifier elimnation.
Th(Q; $\times$) is decidable.and
$(\mathbf{Q}^{+};\trianglelefteq) \leq \left(\mathbb{Q}^{+} \quad \times\right)$ $\$$
and $a \trianglelefteq b \;\rightarrow\; \exists x \; (a \times x = b)$ .We review that$(\mathbf{Q}^{+};\trianglelefteq)$ is axiomatizable and decidable.So this theory is complete.

Tabeles

The first-order theory of Boolean algebras, established by Alfred Tarski in 1940 (found in 1940 but announced in 1949). [2]

TABLE I

Theory of	Proved by	A method of proof
Boolean algebras	by Tarski in 1949	model completeness

The decidability of the structure of rational numbers in different languages is shown in the following tables so that the theories of decidable by
√and, undecidable theories by $\times$is shown.

Table II

L	Structures	The decidablity of the structures
{<}	$\langle Q,<\rangle$	√
{+}	$\langle Q,+\rangle$	√
{×}	$\langle Q,\times\rangle$	√
{<,+,-,0}	$\langle Q,<,+\rangle$	√
{⊑}	$\langle Q,\sqsubseteq\rangle$	√

Acknowledgment
I wish to express my gratitude to my Ph.D. advisor, Professor Saeed Salehi of Research Institute for Fundamental Science (RIFS), University of Tabriz P.O.Box 51666-16471,Bahman 29thBoulevard;Tabriz;Iran
saesal@gmail.com

References

[1] *Clara Trullenque Ortiz, COMPLETE THEORIES OF BOOLEAN ALGEBRAS,Departament de Matematiques Informatica,June 27, 2018.*

[2] *Monk, J. Donald; Mathematical Logic, Springer (1976), ISBN: 9780387901701.*

[3] *Steven Givant, Paul Halmos, Introduction to Boolean Alge- bras,Springer (2009)*

[4] *Burris, Stanley N., and H.P. Sankappanavar, CA Course in Universal Algebra,1981. Springer-Verlag. ISBN 3-540-90578-2 Free online edition.*

[5] *https://plato.stanford.edu/entries/mereology/notes.html1*

[6] *https://www.oxfordbibliographies.com/view/document/obo- 9780195396577/obo-9780195396577-0313.xml*

[7] *https://en.wikipedia.org/wiki/Mereology*

[8] *Boolos, George S. Burgess, John P. Jeffrey, Richard C, Computability and Logic,Cambridge University Press (5th ed. 2007), ISBN: 9780521701464.*

[9] *Givant Steven and Halmos Paul , Introduction to Boolean algebras, Undergraduate Texts in Mathematics. Springer, 2009*

[10] *Salehi, Saeed, On Axiomatizability of the Multiplicative Theory of Numbers, Fundamenta Informaticae ,159:3 (2018) 279-296. DOI: 10.3233/FI-2018-1665*

[11] *Boolos, George S. Burgess, John P. Jeffrey, Richard C, Computability and Logic,Cambridge University Press (5th ed. 2007), ISBN: 9780521701464.*

[12] *Enderton, Herbert B, A Mathematical Introduction to Logic, Academic Press (2nd ed. 2001) , ISBN: 9780122384523.*

[13] *Salehi,Saeed,Axiomatizing Mathematical Theories :Multiplica- tion, in:A.Kamali-Nejad (ed). Proceedings of Frontiers in Mathematical Sciences ,Sharif University of Technology .Tehran , Iran 2012 , pp.165-176.*

[14] *Cegielski, Patrick; The Elementary Theory of the Natural Lattice is Finitely Axiomatizable, Notre Dame Journal of Formal Logic 30:11:988138-150.*

[15] *D. Prawitz, Dag Westerstahl, Logic and Philosophy of Science in Uppsala, Springer Science Business Media, Tir 8, 1392 AP - Philosophy - 614 pages. .*

[16] *Givant Steven and Halmos Paul , Introduction to Boolean algebras, Undergraduate Texts in Mathematics. Springer, 2009*

[17] *Dutilh Novaes, Catarina; Formal Languages in Logic,November 2012, ISBN: 9781107020917.*

[18] *Christopher C Leary, Lars Kristiansen; A Friendly Introduction to Mathematical Logic, Milne Library, 2019.*

[19] *Smorynski, Craig; Logical Number Theory I, Springer-Verlag (1991).*

.