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Solving NP-Hard Problem Using a New Relaxation of Approximate Methods

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***Abstract*---**A new approach has been introduced for solving NP-hardness problem in combinatorial optimization problems. Actully, our study focused on the relationship between the Lagrange method and Penalty method, this paper introduce a new relaxation of the fesible region.Furthermore, NP hard problem has been tested and showed that the Augmented Lagrangian Approach outperformed the Penalty method. Finally, our study focuses on enhancing the theoretical convergence features as well as numerical computing.

***Keywords*---**optimization problems, relaxation, approximate methods, penalty, augmented lagrangian methods.

Introduction

Optimization, in general, is the problem of minimizing or maximizing a function under a set of constraints. Problems with optimization abound. Every for-profit business owner has the challenge of increasing profit while working with restricted resources. In general, this is too broad an issue to be addressed precisely; nevertheless, optimization techniques may be used to successfully address many elements of decision-making [1,2,3] . This involves problem with production, inventories, and machine scheduling, to name a few. Indeed, optimization strategies are used by the vast majority of businesses. Optimization issues, on the other hand, are not restricted to the business sphere. Every time you use your GPS, it solves an optimization problems, namely how to reduce the time it takes to travel between two points. Your municipality may desire to reduce the number of garbage trucks required by determining the most effective route for each vehicle[3,4,5,16]

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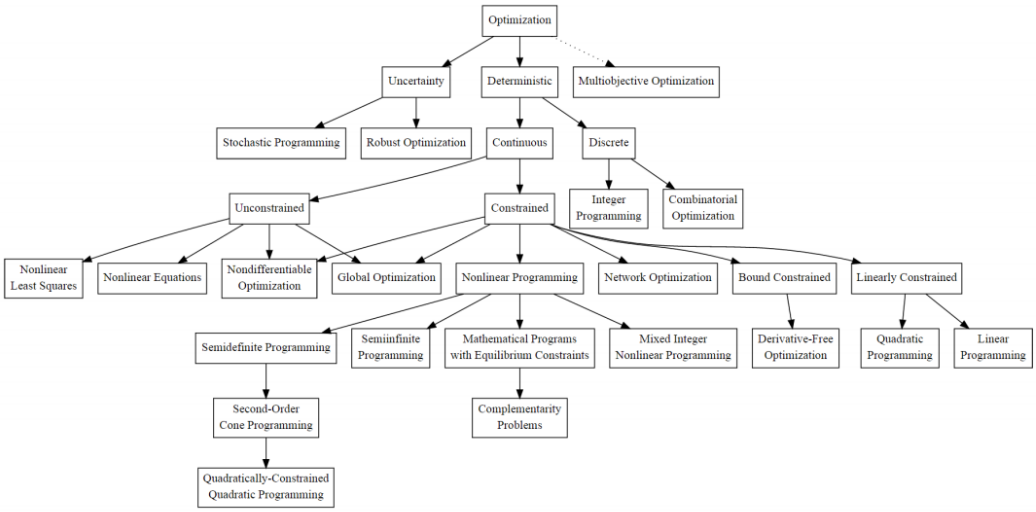


Fig. (1): General scheme for optimization branches

In order to provide effective service to its residents, city planners may need to select where new fire stations should be built. Other examples include how to build a portfolio that maximizes expected return while limiting volatility; how to build a resilient tele-communication network as cheaply as possible; how to schedule flights in a cost-effective manner while meeting passenger demand; and how to schedule final exams using the fewest possible classrooms. A two-step procedure will be followed in order to maximize profit resources utilizing the principle of optimization:

1. Develop a formulation for the optimization problems.
2. Solve the formulation with an appropriate method.

The optimization problem is represented mathematically by a formulation. In your formulations, the different factors that the owner seeks to ascertain are represented as variables (unknowns). The quantity to be maximized will be represented by the goal function. Finally, each problem constraint is represented as a mathematical constraint. Now, given an adequate mathematical formulation, you must build (or use an existing) method to solve the formulation. By algorithm, we mean a finite method (something that can be programmed as a computer program) that takes the formulation as input and returns an assignment of values to each of the variables that satisfies all constraints and maximizes the objective function under these conditions. The variables' values show the best options for determining the parameters that the for-profit business owner wants to know. Figure 1.1 summarizes the two-step method. Finally, we focused our efforts on developing a new relaxation of the feasible region in order to derive the Lagrange method using the Penalty approach.

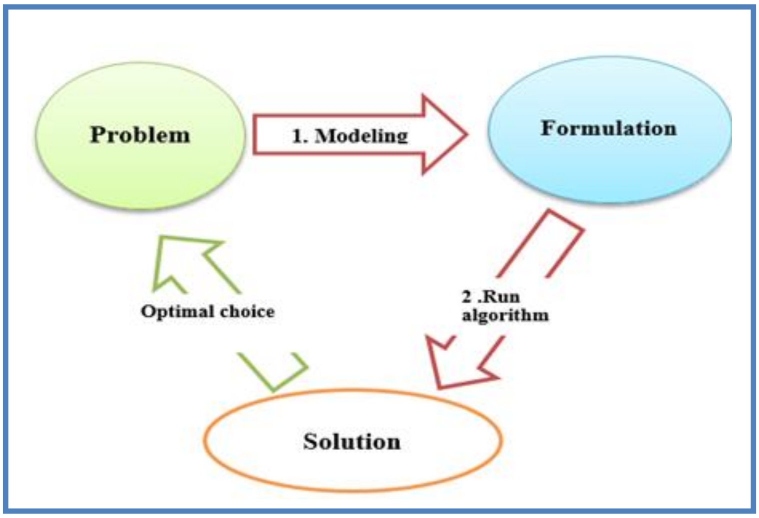


Fig. (2) Optimization challenge modeling

Mathematical programming

A Mathematical Programming or (Optimization problem) can expressed as following [6,7,17]:

$$\begin{cases} \min f(k) \\ subject\ to\ w_i(k) = 0, i = 1, \dots, E \\ k \in K \end{cases} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (1)$$

Where f and w_i defined from K ⊆ R is the set of feasible solutions and the objective function represented by K→R. The standard procedure is to define each problem as a minimization problem and, if necessary, to use the substitution

$$\max\{f(k): k \in K\} = -\min\{-f(k): k \in K\}.$$

An infeasible problem is one for which there is no feasible solution, according to this case, it was agreed to write :

$$\min\{f(k): k \in K\} = +\infty$$

Unbounded problems are defined as those in which f is not constrained from below in K, so according to this case:

$$\min\{f(k): k \in K\} = -\infty$$

The goal of problem (1) is to find an optimal solution, if one exists.

$$k^* \in K\ such\ that\ f(k^*) \leq f(k), \forall\ k \in K.$$

This solution may be not have to be unique.

NP-Hard Problem

There are many different sorts of tough and significant problems that may be categorized based on the solution technique and algorithms employed. If a non-deterministic Turing machine can solve a decision problem in polynomial time, it is labeled as NP. Non-deterministic polynomial time is abbreviated as NP. The non-deterministic Turing machine employs a two-phase algorithm. The first step is based on a non-deterministic estimate of the answer. A polynomial-time deterministic method is used in the second step to verify that the estimate is the correct solution to the problem.

When there is at least one polynomial-time method to solve a decision problems, it is deemed to be in the P-class. By this case, the algorithm's solution time is constrained by a polynomial in *n*, where n is the length of the input. A decision issue is NP-complete if any other NP problem can be reduced to the NP-complete problem in polynomial time by a deterministic Turing computer. Also, an issue is NP-hard, but not necessarily a decision problem, if any other NP problem can be reduced to the NP-hard problem in polynomial time by a deterministic Turing computer. The NP-complete problems are the most difficult in NP because, unless P=NP, a polynomial-time method to solve them is unlikely to exist. If P=NP, however, there may be asymptotically quicker methods for NP problems [1, 2, 3, 6,19].

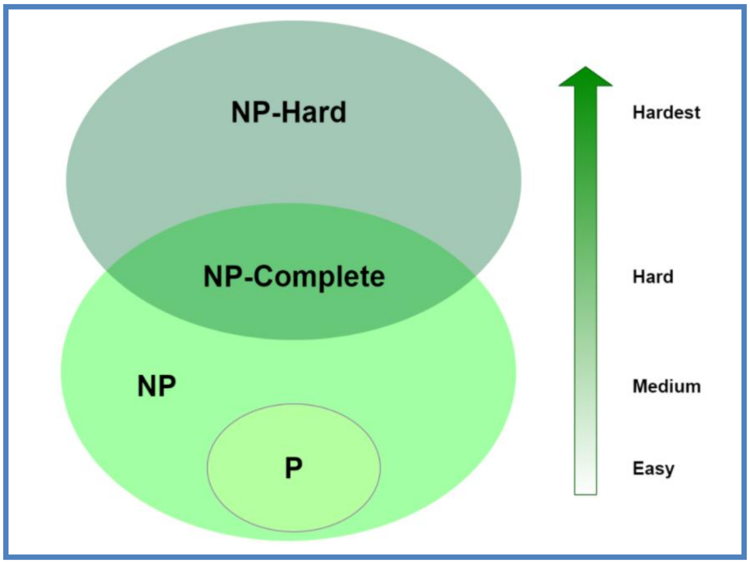


Fig. (3): The Classification of NP-Hardness

Relaxation

As is well known, the majority of real-world graph or network problems are NP-Hard which is hard to solve. In this case, solving a simpler problem to gain estimates or constraints on the initial toughest challenge may be of interest. Consider the optimization problem where f: $R^n \rightarrow R$ and S subset from R^n:

$$\begin{cases} \text{minimize } f(k) \\ subject\ to\ k \in S \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)} \end{cases}$$

The following form is a relaxation of the aforementioned problem:

$$\begin{cases} \text{minimize } f_R(k) \\ subject\ to\ k \in S_R \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)} \end{cases}$$

Where $f_R: R^n \to R$ such that $f_R(k) \leq f(k)$ for all $\text{k} \in S$ and$S \subseteq S_R$. It is obvious that the relaxation's optimum solution f_R^* is a lower constraint on the initial problem's optimal solution.

According to the same concept, the Lagrangian relaxation aims to leverage the underlying network structure of these problems to implement these efficient methods. The Lagrangian Relaxation is a decomposition method:
The problems constraints $S = S_1 \cup S_2$ are divided into two groups: the'easy' constraints S1 and the'hard' constraints S2. The hard restrictions are then eliminated, i.e., S_R =S_1, and the objective function, f_R relies on f and S_2, is transferred.The relaxation problem will be solved since S_R is a collection of‘easy' constraints.The Lagrangian relaxation is also interesting because, in some circumstances, the optimal solution of the relaxed issue is the same as the optimal solution of the original problem. To explain the principle of relaxation, we review this simplified form, Consider we have Hard problem represented by the form and formula below [1,8,9]:

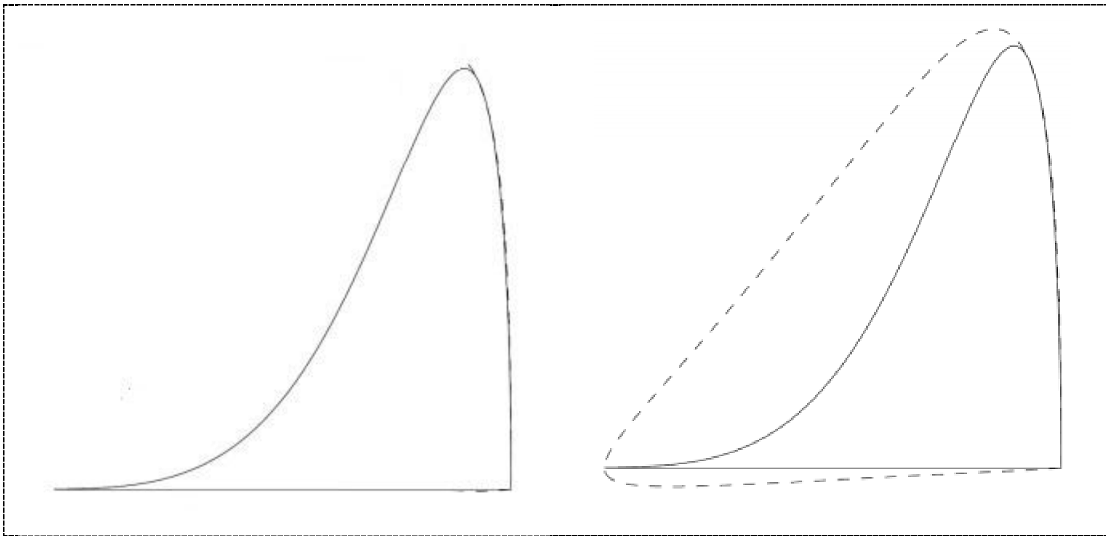
$$P_1 : \min_{k \in K} f(k)$$

When we make relaxation for this problem according to the principle of relaxation, we will get this form by which we can reach a quick and easy solution by applying theories and facts in this regard:

$$P_2 : \min_{k \in Y} f(k), K \subset Y$$

By comparing the diagram of the two functions (P_1 and P_2), we clearly find the concept of the following facts.

$$(Obj.of\ P_2) \leq (Obj.of\ P_2)$$



$P_1 : \min_{k \in K} f(k)$ | $P_2 : \min_{k \in Y} f(k), K \subset Y$

Fig. (4): The problem after and before relaxation

Proposition

If k* is optimal for relaxation and feasible for exact, then k is optimal for exact. [10].

Proof: assume k^*_1 is relaxed optimal and feasible suboptimal for exact problem that is there exist y such that $f(y) < f(K_1^*), y \in K$. But according to the relaxation, $y \in Y$ and in this case k^*_1 is not relaxed optimal, this leads to a contradiction!

Example 4.1

Let the following optimization problem:

$$\begin{cases} \min_z & z_1(z_2 - 1) \\ \text{subject to} & z_1 \geq 1 \\ & z_2 \geq 2 \\ & z_1 z_2 \leq 3 \end{cases}$$

Clearly, this is not a convex problem (objective and a constraint). The relaxation steps for this problem are as follows:
Let $m = z_1 z_2$ then the optimization problem becomes:

$$\begin{cases} \min_{k,y} & m - z_1 \\ \text{subject to} & z_1 \geq 1 \\ & z_2 \geq 2 \\ & m \leq 3 \\ & . \end{cases}$$

$$m - 2z_1 - z_2 + 2 \geq 0$$
$$= (z_1 - 1)(z_2 - 2)$$

Thus, the final constraint ensures that $\neq -\infty$.

Karush-Kuhn-Tucker Conditions

Equality Constrained Optimization [1,11,12]
Assume k^* minimizes the following

$$\text{minmizing } f(k)$$
$$\text{subject to } \mathrm{h}_i(\mathrm{k}) = 0, i = 1, \dots \dots, m$$
$$k \in R^n$$

In this case the following are two conditions are possible:

1- The vector $\nabla h_1(k^*), \nabla h_2(k^*), \dots, \nabla h_m(k^*)$ are linearly dependent
2- There exists a vector μ^*such that

$$\frac{\partial L(k^*, \mu^*)}{\partial k_1} = \frac{\partial L(k^*, \mu^*)}{\partial k_2} = \frac{\partial L(k^*, \mu^*)}{\partial k_3} =, \dots, = \frac{\partial L(k^*, \mu^*)}{\partial k_n} = 0$$

$$\frac{\partial L(k^*, \mu^*)}{\partial \mu_1} = \frac{\partial L(k^*, \mu^*)}{\partial \mu_2} = \frac{\partial L(k^*, \mu^*)}{\partial \mu_3} =, \dots, = \frac{\partial L(k^*, \mu^*)}{\partial \mu_m} = 0$$

For this case let consider the following example.

Example 5.1

Consider the problem

$$\text{minmizing } k_1 + k_2 + k_3^2$$
$$\text{subject to } \mathrm{k}_1 = 1$$
$$k_1^2 + k_2^2 = 1$$

1- The minimum is achieved when $k_1 = 1, k_2 = 0, k_3 = 0$
2- The Lagrangian is:
3-

$$L(k_1, k_2, k_3, \mu_1, \mu_2) = k_1 + k_2 + k_3 + \mu_1(1 - k_1) + \mu_2(1 - k_1^2 - k_2^2)$$

it clear that:

$$\frac{\partial L(1,0,0, \mu_1, \mu_2)}{\partial k_2} = 1 \,, \mu_1, \mu_2$$

And $\nabla k_1(1,0,0) = [1 \ \ 0 \ \ 0]$ and $\nabla k_2(1,0,0) = [2 \ \ 0 \ \ 0]$

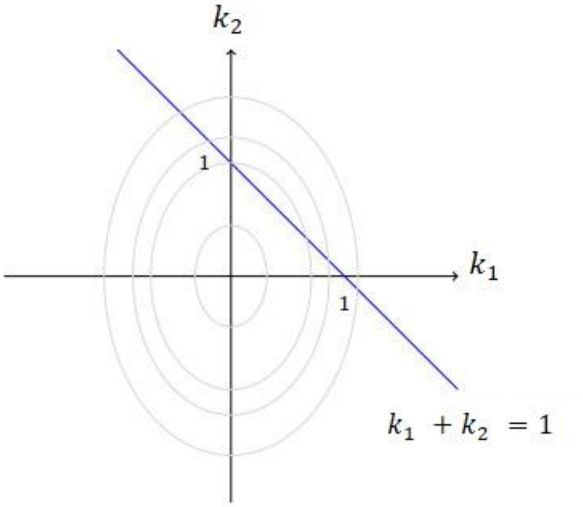


Fig. (5): Application the KKT condition equality constrained optimization

Approximate Method

In optimization, approximation approaches are a very interesting and effective concept. Let $\mu: R^n \to R$ is the minimizing convex function on a convex set K. Approximation approaches aim to replace μ and K with approximation μ^h and K^h . It is noteworthy, only when the approximation is easier than the original problem does the approximation approach work. For each iteration h we seek to find

$$k^{h+1} = \underset{k \in K^h}{argmin}\, \mu^h(k), \ldots\ldots \quad (4)$$

Then, in the following iteration, μ^{h+1} and K^{h+1} are produced by an approximation that is based on the new point k^{h+1}. Many outstanding approximation approaches, such as polyhedral approximation, interior point methods, the augmented Lagrangian method, and the penalty method, are based on this concept. The augmented Lagrangian method is the focus of this research [12,13].

The Main Idea

In this section, the main idea will be the derivation the Augmented Lagrange method based on Penalty method by creating a new relaxation of the feasible region. In the 1970s, this method became popular. Initially, it was known as the multiplier approach. The augmented Lagrangian method is the name given to this method currently. The goal of this technique is to address issues involving restricted optimization. Because both methods add a penalty term to the objective, the augmented Lagrangian method is comparable to the penalty method [14, 15,18]. The augmented Lagrangian method differs simply in that it includes a Lagrange multiplier term. Let we define the standard penalty function or (quadratic penalty function) of problem (1) as follow:

$$P_q(k,\alpha) = f(k) + \frac{1}{2\alpha}\|w(k)\|^2, \alpha > 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

For *a>0* consider *k (a)* be a minimizer of Pq (k,a). This methods provide a sequence of solutions which are infeasible. Every iterate k (a) is either a locally optimum solution or infeasible for the problem (1). These approaches are advantageous because they may employ strong methods for addressing unconstrained problems despite their relative simplicity. They are inefficient in reality, despite having a strong theoretical foundation, since the series of unconstrained problems does not yield a precise solution. As shown in Figure (3), the penalty parameter of the penalty function method must be go to zero.

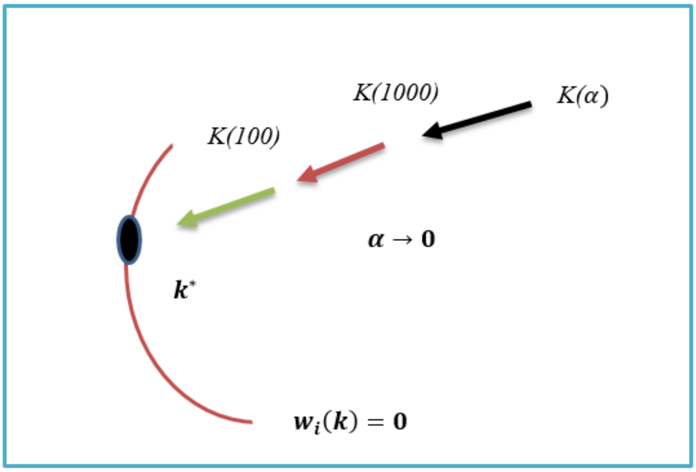


Figure (6): In penalty methods, *k(a)* approaches *k* as a → 0 .*

In order to derive the augmented Lagrangian method, we will change the constraint $w_i(k) = 0$ in above method for problem (1) to the constraint $w_i(k) + \frac{\alpha}{\beta}$ as shown in the figure (4). As a result, we have the problem

$$\begin{cases} \min f(k) \\ subject\ to\ w_i\,(k) + \frac{\alpha}{\beta} = 0, i = 1, \ldots, E \\ k \in K \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

We can derive the augmented Lagrangian method by using the penalty approach on problem (6) as follows.

$$argmin_x f(k) + \frac{1}{2\alpha}\left(w(k) + \frac{\alpha}{\beta}\right)^T (w(k) + \frac{\alpha}{\beta})$$

which can be expanded to

$$argmin_x f(k) + \frac{1}{2\alpha}(w(k)^T w(k) + \frac{2\alpha}{\beta^T} w(k) + \frac{\alpha^2}{\beta^T \beta})$$

and can simplifies to

$$argmin_x f(k) + \frac{1}{\beta^T} w(k) + \frac{1}{2\alpha} \|w(k)\|^2$$

As a result, the augmented Lagrangian function is

$$L(k, \beta, \alpha) = f(k) + \frac{1}{\beta^T} w(k) + \frac{1}{2\alpha} \|w(k)\|^2.$$

In the 1970s, the augmented Lagrangian method became popular and it was introduced by Hestenes. Initially, it was known as the multiplier method.

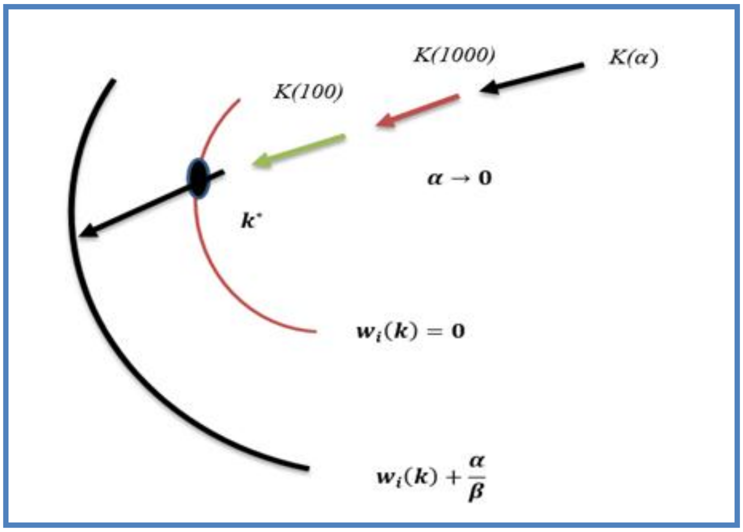


Figure (7): We modify $w_i(k) = 0$ to $w_i(k) + \frac{\alpha}{\beta}$ in the augmented Lagrangian technique to get k^* with a finite value of α.

Numerical Results

In table 8.1, the types of graphs (w01, w05, w09), in (Figure 8, 9, 10), which shows is tested according to the solution methods (Penalty method, Augmented Lagrange method), where the results show that the combined algorithm has better convergence properties than the other two methods. The combination method technique, which was faster than the augmented Lagrange function call with the penalty method, was used to reach this bound. Moreover, the penalty approach introduced a speed smaller than convergence to reach the same bound, as seen in all of the graphs. Finally, our built-in algorithm's convergence success is due to the augmented Lagrangian.

Our results					
Graphs	**goal bound**	**Optimal value**	**Penalty**	**Aug**	**Mixed**
w01-100.0	651.45771	651	1780	1334	880
w01- 100.1	719.57932	719	1618	1972	1140
w01- 100.2	692.56110	676	912	463	547
w01- 100.3	814.92739	813	1895	1353	904

w01- 100.4	668.00791	668	863	737	736
w01- 100.5	645.17064	643	1695	1512	1175
w01- 100.6	654.04204	654	1114	1484	1106
w01- 100.7	728.28042	725	1695	1945	1185
w01- 100.8	721.00644	721	740	691	808
w01- 100.9	729.02160	729	1170	837	855
w05- 100.0	1739.41303	1646	641	535	367
w05- 100.1	1668.85603	1606	636	512	444
w05-100.2	1958.78729	1902	795	348	394
w05- 100.3	1714.91680	1627	731	820	360
w05- 100.4	1641.78351	1546	467	450	384
w05- 100.5	1676.97781	1581	642	403	403
w05- 100.6	1544.62023	1479	968	879	407
w05- 100.7	2032.24223	1987	907	812	445
w05- 100.8	1409.28063	1311	681	736	356
w05- 100.9	1791.86734	1752	758	993	903
w09- 100.0	2235.05951	2121	693	703	524
w09- 100.1	2264.55563	2096	633	575	953
w09- 100.2	2881.52059	2738	623	426	595
w09- 100.3	2133.05091	1990	477	638	316
w09- 100.4	2155.59192	2033	725	745	542
w09- 100.5	2455.72668	2433	1364	589	618
w09- 100.6	2282.87760	2220	909	406	549
w09- 100.7	2356.16199	2252	653	425	512
w09- 100.8	1925.04385	1843	1039	697	659
w09- 100.9	2162.28285	2043	968	438	503

Table 1: On many graphs from the Biq Mac library, the faction call augmented Lagrangian and penalty methods were measured.

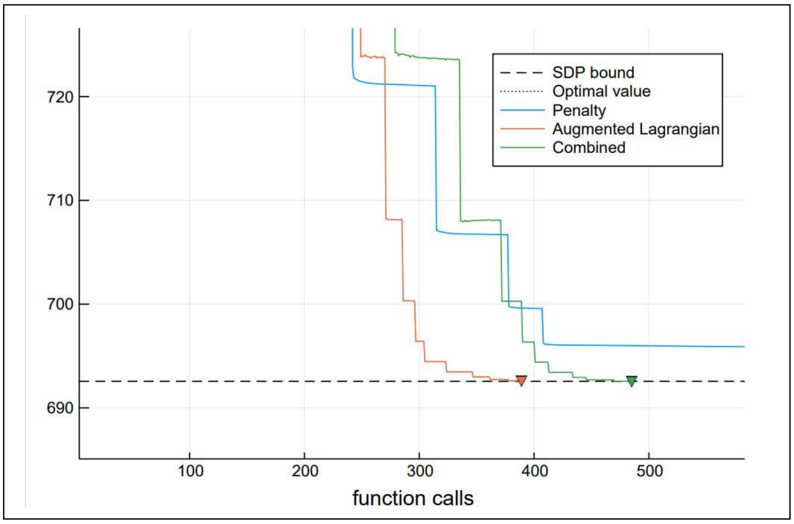


Figure (8): On graph w01- 100, bounds versus function calls for inequality augmented Lagrangian, penalty methods, and combination technique.

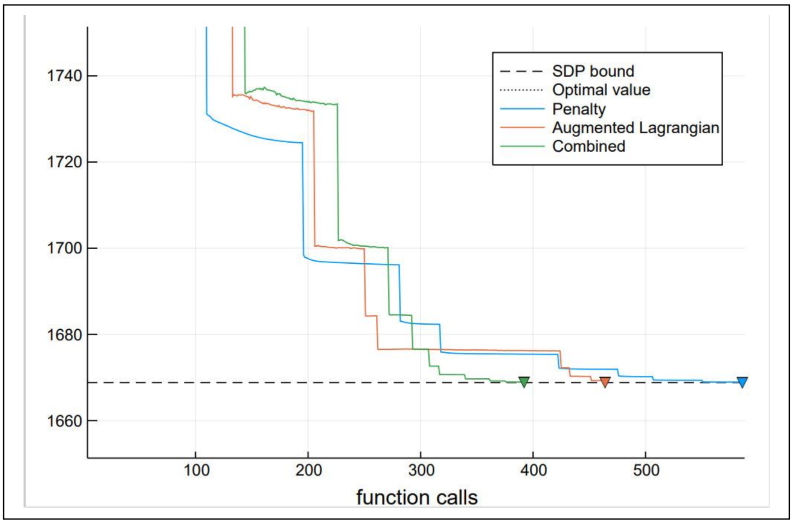


Figure (9): On graph w05- 100, bounds versus function calls for inequality augmented Lagrangian, penalty methods, and combination technique.

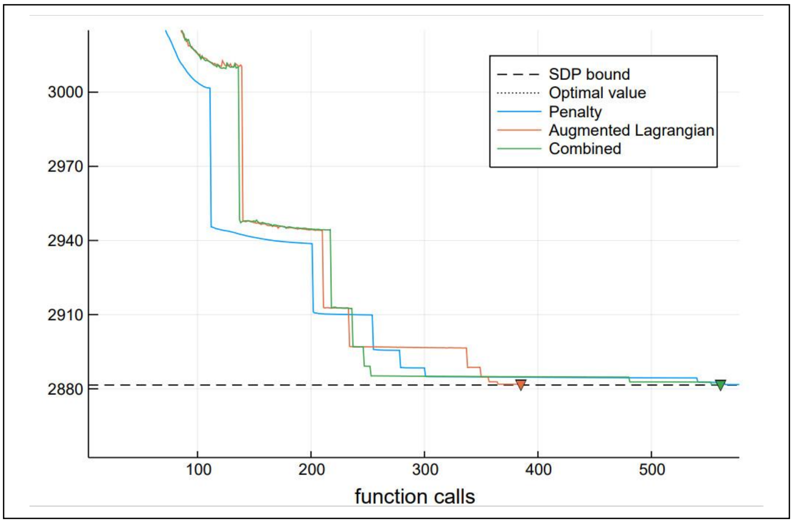


Figure (10): On graph w05- 100, bounds versus function calls for inequality augmented Lagrangian, penalty methods, and combination technique.

Conclusion

A novel technique for tackling NP-hardness problems in combinatorial optimization problems has been developed. In fact, in this research the focus was on the connection between the Lagrange and Penalty methods. Sides, this work introduces a novel relaxation of the feasible region. Furthermore, the augmented lagrangian methods outperformed the Penalty approach when applied to the NP hard problem. Finally, the numerical results gave a great support to the research objective and to improve theoretical convergence as well as numerical computation.

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