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Solving the second kind linear volterra integrodifferential equations using the complex SEE integral transform

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***Abstract*---**As an important type of integral equation, Volterra integral equations snatch the focus of many scientists and mathematicians to provide approximate or exact solutions to such equations. The integral transform capability of providing an algebraic solution to the integral equations led the mathematical community to lean heavily on them to solve those kinds of equations, including the Volterra integrodifferential equations of the second kind. This paper uses the complex SEE integral transform to find the exact solution to the second kind linear Volterra integrodifferential equation. The capability and efficiency of complex SEE integral transform in providing an exact solution with the minimum number of computations possible are demonstrated via practical applications.

***Keywords*---**complex SEE, integral transform, inverse complex, SEE integral transform, integrodifferential equations, volterra.

Introduction

The prospect of linear and nonlinear integral equations where the differential and integral operators appear in the same equation, had been presented by Vito Volterra [1]. Volterra integrodifferential equations have been used in a variety of

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fields, including epidemic spread, population flow, and semi-conductors; they have had a significant impact on physics, space, and many other fields [2].

The vast importance of Volterra integrodifferential equations encouraged mathematicians to provide as many as possible methods to solve them [3-7]. Integral transforms stand out as an adequate solving method for Volterra equations because of their ability to provide an algebraic solution to the integral equations [8-11]. As the focus of this study, the second kind linear Volterra integral transform has been solved by many integral transforms [12-14]. In 2021, Eman A. Mansour et al. introduced a new integral transform under the name "Complex (Sadiq-Emad-Eman) SEE" integral transform [15,16]. The Complex SEE integral transform will be used in this work to find the exact solution to the linear second kind Volterra integrodifferential equation and will be proven by some practical applications.

Fundamental Concepts Of The Complex See Integral Transform

The complex SEE (Sadiq-Emad-Eman) integral transform of the function $g(t)$ is defined as [15,16]:

$$S^c\{g(t)\} = \frac{1}{v^n}\int_{t=0}^{\infty} e^{-ivt} g(t)\, dt = T(iv) \ \ , \ t \geq 0 \ \ , \ l_1 \leq v \leq l_2 \ , \ n \in \mathbb{Z}, i \equiv \text{imaginary number} .$$

Linearity Property of the Complex SEE Transform

$$S^c\{\alpha f(t) + \beta g(t)\} = \alpha S^c\{f(t)\} + \beta S^c\{g(t)\}.$$

Where α and β are arbitrary constants [15,16].

The Complex SEE Transform for Some Functions

The complex SEE integral transform is applied to some basic functions [15,16] as:

- $S^c[C] = -\frac{iC}{v^{n+1}}$, $i \equiv complex\ number\ and\ i^2 = -1.$
- $S^c[t] = -\frac{1}{v^{n+2}}.$
- $S^c[t^2] = \frac{(2!)i}{v^{n+3}}.$
- In general: $S^c[t^m] = \frac{(-1)^m (i)^{m-1}\, m!}{v^{n+m+1}}$ $, m \in \mathbb{Z}^+.$
- $S^c[e^{ct}] = -\frac{1}{v^n}\left[\frac{c}{c^2+v^2} + i\frac{v}{c^2+v^2}\right]$, $c \equiv constant$.
- $S^c[\sin(ct)] = \frac{-c}{v^n(v^2-c^2)}$.
- $S^c[\cos(ct)] = \frac{-iv}{v^n(v^2-c^2)}$.
- $S^c[\sinh(ct)] = \frac{-c}{v^n(v^2+c^2)}$.
- $S^c[\cosh(ct)] = \frac{-iv}{v^n(v^2+c^2)}$.

The Complex SEE Transform for the Derivatives of the Function $\boldsymbol{u}(\boldsymbol{x})$
If $T(iv)$ is the complex SEE integral transform for the function (x) , then $T(iv) = S^c\{u(x)\}$. The complex SEE transform for $u(x)$ derivatives is as follows [15,16]:

- $S^c\{u'(x)\} = \frac{-u(0)}{v^n} + \mathrm{i}vT(iv)$.
- $S^c\{u''(x)\} = \frac{-u'(0)}{v^n} - \frac{iu(0)}{v^{n-1}} + v^2T(iv)$.
- In general: $S^c\left[u^{(m)}(x)\right] = \frac{1}{v^n}\left[-u^{(m-1)}(0) - ivu^{(m-2)}(0) - (iv)^2u^{(m-3)}(0) - \cdots - (iv)^{m-1}u(0)\right] + (iv)^mT(iv)$.

The Convolution Property for the Complex SEE Transform

If $S^c\{G(t)\} = T_1(iv)$ and $S^c\{H(t)\} = T_2(iv)$, then $S^c\{G(t) * H(t)\} = v^n\, S^c\{G(t)\} * S^c\{H(t)\} = v^n\, T_1(iv).T_2(iv)$ [15,16].

Fundamental Concepts Of The Inverse Complex See Integral Transform

If $S^c\{u(t)\} = T(iv)$, then $u(t)$ is called the inverse of the complex SEE transform for $T(iv)$ and it is defined by [15,16]:
$f(t) = S^{c^{-1}}\{F(iv)\} = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{F(iv)}{v^n} e^{ivt} dv$, where δ is a posetive fixed number, $t \geq 0$, i is a complex number $(i^2 = -1)$, $n \in \mathbb{Z}$.

The Inverse Complex SEE Transform for Some Functions
The inverse complex SEE integral transform is applied to some basic functions [15,16] as:

- $S^{c^{-1}}[C] = -\frac{iC}{v^{n+1}}\,, C \equiv Constant.$
- $S^{c^{-1}}[t] = -\frac{1}{v^{n+2}}.$
- $S^{c^{-1}}[t^m] = \frac{(-1)^m(i)^{m-1}\, m!}{v^{n+m+1}}, m \in \mathbb{Z}^+.$
- $S^{c^{-1}}[e^{ct}] = -\frac{1}{v^n}\left[\frac{c}{c^2+v^2} + i\frac{v}{c^2+v^2}\right]\,, c \equiv constant\,.$
- $S^{c^{-1}}[\sin(ct)] = \frac{-c}{v^n(v^2-c^2)}\,.$
- $S^{c^{-1}}[\cos(ct)] = \frac{-iv}{v^n(v^2-c^2)}\,.$
- $S^{c^{-1}}[\sinh(ct)] = \frac{-b}{v^n(v^2+c^2)}\,.$
- $S^{c^{-1}}[\cosh(ct)] = \frac{-iv}{v^n(v^2+c^2)}\,.$

Solving The Second Kind Linear Volterra Integrodifferential Equations by The Complex See Integral Transform

Volterra integrodifferential equations occur when an initial value problem is transformed into an integral equation using the Leibnitz rule1,2. The linear Volterra integrodifferential equation of second kind is defined as12-14:

$$f^{(m)}(x) = g(x) + \lambda \int_{t=0}^{x} f(t)k(x,t)dt \qquad (1)$$

The m^{th} derivative of the unknown function $f(x)$ with respect to the variable (x) is $f^{(m)}(x)$. Hence, Eq.(1) contains integration and differentiation, then it is necessary to define the initial conditions $f(0), f'(0), f''(0), \dots, f^{(m-1)}(0)$ for it.

Assuming Eq. (1) kernel is a difference kernel, which can be expressed by difference $(x-t)$, it is possible to write the equation of the linear second kind Volterra integrodifferential equation as:

$$\left.\begin{aligned} u^{(m)}(x) &= f(x) + \lambda \int_{t=0}^{x} k(x-t)u(t)dt \\ \text{with conditions } \ u(0) &= a_0 \ , \ u'(0) = a_1 \ , \ \dots, u^{(m-1)}(0) = a_{m-1} \end{aligned}\right\} \qquad (2)$$

To find the exact solution to Eq.(2), the complex SEE integral transform is applied to both sides of it as:

$$\frac{1}{v^n}\left[-u^{(m-1)}(0) - iv\, u^{(m-2)}(0) - (iv)^2\, u^{(m-3)}(0) - \cdots - (iv)^{m-1}\, u(0)\right] + (iv)^m S^c\{u(x)\} = S^c\{g(x)\} + \lambda\, S^c\left\{\int_{t=0}^{x} k(x,t)u(t)dt\right\} \qquad (3)$$

$$\Rightarrow (iv)^m\, S^c\{u(x)\} = \frac{a_{m-1}}{v^n} + \frac{iv\, a_{m-2}}{v^n} + \cdots + \frac{(iv)^{m-1}\, a_0}{v^n} + S^c\{g(x)\} + \lambda\, S^c\left\{\int_{t=0}^{x} k(x,t)u(t)dt\right\} \qquad (4)$$

Using the complex SEE transform convolution property on Eq.(4):

$$S^c\{u(x)\} = \frac{a_{m-1}}{v^n\,(iv)^m} + \frac{iv\, a_{m-2}}{v^n\,(iv)^m} + \cdots + \frac{(iv)^{m-1} a_0}{v^n\,(iv)^m} + \frac{S^c\{g(x)\}}{(iv)^m} + \frac{\lambda}{(iv)^m} v^n . S^c\{k(x)\}. S^{c\{u(x)\}} \qquad (5)$$

Applying the inverse complex SEE transform to both sides of Eq.(5):

$$u(x) = a_{m-1}.\frac{x^{m-1}}{(m-1)!} + a_{m-2}.\frac{x^{m-2}}{(m-2)!} + \cdots + a_0 + S^{c^{-1}}\left\{\frac{1}{(iv)^m} S^c\{g(x)\}\right\} + \lambda S^{c^{-1}}\left\{\frac{v^n}{(iv)^m} S^c\{k(x)\}. S^c\{u(x)\}\right\} \qquad (6)$$

Equation (6) represents the exact solution to the linear Volterra integrodifferential equation of the second kind in Eq.(2).

Examples

The efficiency of the complex SEE integral transform in calculating the exact solution to the second kind linear Volterra integrodifferential equations is demonstrated through some applications in this section.

Example (1)

Consider the second kind linear Volterra integrodifferential equation with the following conditions: $u'(x) = 2 + \int_{t=0}^{x} u(t)dt$, $with\, u(0) = 2.$ (7)

Applying the complex SEE transform to both sides of Eq. (7) and using the provided conditions, gives: $\frac{-u(0)}{v^n} + ivS^c\{u(x)\} = S^c\{2\} + S^c\{\int_{t=0}^{x} u(t)dt\}$,

$\Rightarrow ivS^c\{u(x)\} = \frac{2}{v^n} + \left(\frac{-2i}{v^{n+1}}\right) + S^c\{\int_{t=0}^{x} u(t)dt\}$, (8)

Now, applying the complex SEE transform convolution Property to Eq.(8), provides:

$ivS^c\{u(x)\} = \frac{2}{v^n} - \frac{2i}{v^{n+1}} + v^n \left(\frac{-i}{v^{n+1}}\right) S\{u(x)\}$, $\Rightarrow \left(iv + \frac{i}{v}\right) S^c\{u(x)\} = \frac{2}{v^n} - \frac{2i}{v^{n+1}}$, $\Rightarrow \left(\frac{iv^2+i}{v}\right) S^c\{u(x)\} = \frac{2}{v^n} - \frac{2i}{v^{n+1}}$.

Then: $S^c\{u(x)\} = \frac{2v}{v^n(iv^2+i)} - \frac{2iv}{v^{n+1}(iv^2+i)} = \frac{(2v-2i)}{v^n(iv^2+i)} = \frac{-2}{v^n}\left[\frac{1}{v^2+1} + \frac{iv}{v^2+1}\right]$, (9)

Applying the inverse complex SEE transform to both sides of Eq.(9), obtains: $u(x) = 2e^x$. (10)

The concluded result from Eq.(10) is the exact solution to Eq.(7).

Example (2)

Consider the second kind linear Volterra integrodifferential equation with following the conditions:

$$\left.\begin{aligned} u''(x) &= 1 + \int_{t=0}^{x} u(t)(x-t)dt \\ with \quad u(0) &= 1 \quad and \quad u'(0) = 0 \end{aligned}\right\} \qquad (11)$$

Applying the complex SEE transform into both sides of Eq.(11) with the provided initial conditions, gives: $\frac{-u'(0)}{v^n} - i\frac{u(0)}{v^{n-1}} - v^2T(iv) = S^c\{1\} + S^c\{\int_{t=0}^{x} u(t)(x-t)dt\}$, $\Rightarrow \frac{-i}{v^{n-1}} - v^2S^c\{u(x)\} = \frac{-i}{v^{n+1}} + S^c\{\int_{t=0}^{x} u(t)(x-t)dt\}$, $\Rightarrow -v^2S^c\{u(x)\} = \frac{i}{v^{n-1}} - \frac{i}{v^{n+1}} + S^c\{\int_{t=0}^{x} u(t)(x-t)dt\}$, $\Rightarrow v^2S^c\{u(x)\} = \frac{i}{v^{n+1}} - \frac{i}{v^{n-1}} - S^c\{\int_{t=0}^{x} u(t)(x-t)dt\}$. (12)

Applying complex SEE transform convolution property of into Eq.(12), provides:

$S^c\{u(x)\} = \frac{i}{v^{n+3}} - \frac{i}{v^{n+1}} + \frac{v^n}{v^2}\left(\frac{1}{v^{n+2}}\right) S^c\{u(x)\}$, Then $\left(1 - \frac{1}{v^4}\right) S^c\{u(x)\} = \frac{i}{v^{n+3}} - \frac{i}{v^{n+1}}$, $\Rightarrow \left(\frac{v^4-1}{v^4}\right) S^c\{u(x)\} = \frac{i}{v^{n+3}} - \frac{i}{v^{n+1}}$,

$S^c\{u(x)\} = \frac{iv^4}{v^{n+3}(v^4-1)} - \frac{iv^4}{v^{n+1}(v^4-1)} = \frac{iv-iv^3}{v^n(v^4-1)} = \frac{-iv\,[-1+v^2]}{v^n(v^2+1)(v^2-1)}$.

Then, $S^c\{u(x)\} = \frac{-iv}{v^n(v^2+1)}$. (13)

Applying the inverse complex SEE transform to both sides of Eq.(13), obtains:

$$u(x) = S^{-1}\left\{\frac{-iv}{v^n(v^2+1)}\right\} = \cosh(x)\,. \quad (14)$$

The concluded result from Eq.(14) is the exact solution to Eq.(11).

Conclusions

This work introduced the complex SEE integral transform as an effective and elegant solution method to one of the most important integrodifferential equations, the linear Volterra integrodifferential equation of the second kind. The applications that are discussed in the application section of this work have proven the phenomenal capability of the complex SEE integral transform in finding the exact solution to the second kind linear Volterra integrodifferential equation, which can be used as a milestone in future usage of the complex SEE integral transform in solving integrodifferential equations in general.

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