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Generalization of complex Al-Tememe transformation and its applications

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***Abstract*---**The increasing need for integral transforms to solve the problems of applications in many scientific fields has led to the relentless pursuit of suggesting new integral transforms. An innovative generalization of the Complex Al-Tememe integral transformation is introduced and discussed thoroughly in this work, where the properties of the proposed transformation and its inverse are exposed by utilizing it in some fundamental functions and the efficiency of the generalized Complex Al-Teme integral transform is demonstrated by using it in solving Euler and multiplication ordinary differential equations with variable coefficients.

***Keywords*---**complex Al-Tememe, integral transform, inverse complex, Al-Tememe transform, Euler's equation, generalization complex Al-Tememe transform.

Introduction

For over two centuries, mathematicians' successes in providing multiple integral transforms were the reason that paved the way for the flourishing of knowledge in many scientific, physical, and engineering fields [1-5]. The integral transform's beginnings, as well as the Fourier and Laplace transforms, may be traced back to P.S. Laplace's (1749-1827) notable work on applied mathematics in the 1780s [2,3]. The Laplace or Fourier transform methods are virtually the same as modern

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operational calculus since they are based on a solid mathematical foundation. Some integral transforms, such as Hilbert, Stieltjes, and Hankel transforms, are commonly used in mathematics, science, and engineering. These transforms can be used in solving beginning and boundary value problems, including but not limited to partial and ordinary differential equations [6,7].

The Al-Tememe and Complex Al-Tememe Transforms are considered among the recently emerged transforms that have considerably impacted the scientific world by employing them in handling many applications in various fields [8-12]. This work introduces a generalization of the Complex Al-Tememe transform as a new integral transform where the properties and applications of the proposed transform on general functions and utilizing it in solving specific examples are presented.

Preliminaries Concepts

This section presents some basic mathematical principles.

Definition (I)

Let g(x) be a function defined on the period (a, b), the integral transform (denoted by $G(s)$) for the function $g(x)$ can be defined as follows [2,3]:

$$G(s) = I\{g(x)\} = \int_{x=a}^{b} k(s,x)g(x)dx.$$

For the convergent integral, $k(s,x)$ is the transform's kernel, which is a two-variables function, and (a, b) is the interval of the transform, where a and b are real values.

Definition (II)

The equation: $a_0 x^n y^{(n)} + a_2 x^{n-1} y^{(n-1)} + \cdots \ldots . + a_{n-1} xy' + a_n y = f(x)$.
where $a_0, a_1, \ldots a_n$ are constants and $f(x)$ is a function for the variable x, is called the ''Euler's equation'' [13].

Definition (III)

For $f(x)$ as a piecewise continuous function on the partitionable interval $[a, b]$, where $[a, b]$ divisions by a finite set of points $a = x_0 < x_1 < \cdots < x_n = b$ such that [14]:

On each subinterval $(x_1, x_i + 1)$, $f(x)$ is continuous, where: $i = 0,1,2, \ldots, n-1$.
There is a jump discontinuities in $f(x)$ at x_i, thus: $|\lim_{x \to x_i} + f(x)| < \infty. i = 0,1,2, \ldots, n-1; |\lim_{x \to x_i} - f(x)| < \infty, i = 0,1,2, \ldots, n$.

Complex Al-Tememe Integral Transform

The Complex Al-Tememe integral transform for the function $g(x)$ is defined as [11]:

$T^c\{g(x)\} = \int_{x=1}^{\infty} x^{-iP}\, g(x)dx = F(iP)\ , x > 1.$
Such that the above integral is convergent in the interval [1,∞)
Where P is a positive constant and x^{iP} is a kernel of transform i=$\sqrt{-1}$

The Complex Al-Tememe transform for some functions is:

- $T^c(1) = \frac{-1}{1+P^2} - \frac{P}{1+P^2}\, i.$
- $T^c(x^n) = \frac{-(n+1)}{P^2+(n+1)^2} - \frac{P}{P^2+(n+1)^2}\, i, n \equiv$ constant.
- $T^c(\ln x) = \frac{1-P^2}{(P^2+1)^2} + \frac{2P}{(P^2+1)^2}\, i$
- $T^c(\sinh a \ln x) = \frac{-a[(P^2+1)-a^2]}{(P^2+1)^2+a^4} + \frac{2aP}{(P^2+1)^2+a^4}\, i,\ a \equiv$ constant.
- $T^c(\cosh a \ln x) = \frac{-(P^2+1)+a^2}{(P^2+1)^2+a^4} - \frac{P\left((P^2+1)+a^2\right)}{(P^2+1)^2+a^4}\, i,\ a \equiv$ constant.
- $T^c(\sin a \ln x) = \frac{-a[(P^2-1)-a^2]}{(P^2+1)^2+a^4} + \frac{2aP}{(P^2+1)^2+a^4}\, i,\ a \equiv$ constant.
- $T^c(\cosh a \ln x) = \frac{[(P^2-1)-a^2]}{(P^2+1)^2+a^4} - \frac{P\left((p^2+1)-P^2\right)}{(P^2+1)^2+a^4}\, i,\ a \equiv$ constant.

The Generalization Of Complex And Inverse Complex Al-Tememe Integral Transform

Definition (IV)

The general Complex Al-Tememe integral transform for the function $f(x)$ for $x > 1$ is defined by:

$$\mathcal{T}_g^c[f(x)] = \int_1^{\infty} x^{-iq(p)}\, f(x)dx = F(iq(p)).$$

Such that this integral is convergent in [1,∞) ,$q(p)$ is a function of parameter p, and $x^{-i\,q(p)}$ is the kernel of this transform, $i = \sqrt{-1}$.

Property (I): The Linearity Property of the Generalization of the Complex Al-Tememe Integral Transform

The general Complex Al-Tememe transformation is linear as:

$$T_g^c(Af(x) \pm \beta g(x)) = AT_g^c(f(x)) \pm \beta\, T_g^c\ (g\ (x)).$$

Where A and β are constants, the function $f(x)$ and $g(x)$ are defined when $x\ > 1$.
Proof

$$T_g^c(Af(x) \pm \beta\ g(x) = \int_1^{\infty} x^{-iq(p)}\ (Af(x) \pm \beta g(x))d\ x.$$
$$= \int_1^{\infty}\ x^{-iq(p)}\ A\, f(x)dx \pm\ \int_1^{\infty} x^{-iq(p)}\beta g(x)d\ x\ ,$$
$$= A\ \int_1^{\infty} x^{-iq(p)}\ f(x)d\ x\ \pm \beta \int_1^{\infty} x^{-iq(p)}\ g(x)dx\ ,$$
$$= A\ T_g^c\ (f(x)) \pm\ \beta T_g^c\ (g(x)).$$

Definition (V): The Inverse of the Generalization of the Complex Al-Tememe Integral Transform

If $T_g^c\,[f(x)] = F(i\,q(p))$ represent a general Complex Al-Tememe transformation of $f(x)$, then f(x) is said to be the inverse of the general Complex Al-Tememe transform and it can be written as: $f(x) = T_g^{c-1}[F(i\,q(p)]$.

Property (II): The Linearity Property of the Inverse of the Generalization of the Complex Al-Tememe Integral Transform

The inverse of T_g^c is linear that as:

$T_g^{c-1}[AF_1(i\,q(p)) \pm \beta F_2(i\,q(p)) = A\,T_g^{c-1}([F_1(iq(p)] \pm \beta\,T_c^{c-1}(F2(i\,q(p)],$ A and β are arbitrary constants.

Proof

$T_g^c[Af_1(x) \pm Bf_2(x)] = \mathrm{A}T_g^c(f_1(x)) \pm BT_g^c\,(f_2(x)),$
$= Af_1(iq(p)) \pm Bf_2\,(iq(p)),$
$= Af_1\,(x) \pm \beta f_2(x) = T_g^{c-1}[\mathrm{A}f_1(iq(p)) \pm\,\beta f_2(iq(p))],$
$=\mathrm{A}T_g^{c-1}\,[\,(f_1(iq(p))] \pm \beta T_g^{c-1}[(f_2\,(iq(p))],$
$=T_g^{c-1}[Af_1(iq(p)) \pm\,\beta f_2(i\,q(p))].$

Proposition (I)

If $T_g^c[f(x)]\; = F(i\,q(p))$ and a is any constant, then
$T_g^c\,[x^{-a}f(x)] = F(i\,q(p) + a).$

Proof

$T_g^c\big(f(x)\big) = \int_1^\infty x^{-\,i\,q(p)} f(x)d\,x \;=\; F\big(i\,q(p)\big)$,
$T_g^c\,[x^{-a}f(x)] = \int_1^\infty x^{-i\,q(p)}\,x^{-a}f(x)d\,x$,
$= \int_1^\infty\, x^{-(i\,q(p)+a)}\,f(x)d\,x \;= F(i\,q(p) + a)$.

Applying The Generalization Of Complex Al-Tememe Integral Transform Into Fundamental Functions

$$T_g^c(1) = \frac{-1}{1+(q(p))^2} - \frac{q(p)}{1+\big(q(p)\big)^2}\,i\,, i = \sqrt{-1} \text{ and } q(p) > 1.$$

Proof

$T_g^c\,(1) = \int_1^\infty x^{-i\,q(p)}\,f(x)dx,$
$= \int_1^\infty x^{-i\,q(p)}\,dx = \frac{x^{-iq(p)+1}}{-iq(p)+1}\Big|_1^\infty ,$
$= 0 - \frac{1}{-iq(p)+1} = \frac{1}{-1+iq(p)} \cdot \frac{-1-iq(p)}{-1-iq(p)} = \frac{-1-iq(p)}{1+\big(q(p)\big)^2}$,
$= \frac{-1}{1+(q(p))^2} - \frac{p}{1+(q(p))^2}\,i.$

$$T_g^c(x^n) = \frac{-(n+1)}{(q(p))^2+(n+1)^2} - \frac{q(p)}{(q(p))^2+(n+1)^2}\,i\ , \text{ where } n \in \mathbb{R} \text{ and } i = \sqrt{-1}\,.$$

Proof

$T_g^c(x^n) = \int_1^\infty x^{-iq(p)} x^n dx = \int_1^\infty x^{n-i\,q(p)} dx = \frac{x^{n-i\,q(p)+1}}{n-i\,q(p)+1}|_1^\infty = 0 - \frac{1}{n-i\,q(p)+1} = \frac{1}{i\,q(p)-(n+1)} \cdot \frac{-i\,q(p)-(n+1)}{-i\,q(p)-(n+1)} = \frac{-i\,q(p)-(n+1)}{(q(p))^2+(n+1)^2} = \frac{-(n+1)}{(q(p))^2+(n+1)^2} - \frac{q(p)}{(q(p))^2+(n+1)^2}.$

$T_g^c(ln(x)) = \frac{1-(q(p))^2}{((q(p))^2+1)^2} + \frac{2\,q(p)}{((q(p))^2+1)^2} i\,, x > 0 \; and \; i\sqrt{-1}.$

Proof

$T_g^c\,(ln\,x) = \int_1^\infty x^{-iq(p)}\, lnx\; dx\;,$

$= \ln x \cdot \frac{x^{-i\,q(p)+1}}{-i\,q(p)+1}|_1^\infty - \int_1^\infty \frac{x^{-i\,q(p)+1}}{-i\,q(p)+1} \cdot \frac{1}{x}\; dx\;,$

$= 0 - 0 - \frac{1}{-i\,q(p)+1}\left[\int_1^\infty x^{-i\,q(p)}\; dx\right],$

$= \frac{1}{i\,q(p)-1}\left[\frac{x^{-i\,q(p)+1}}{-i\,q(p)+1}\right]_1^\infty,$

$= \frac{1}{i\,q(p)-1}\left[0 + \frac{1}{i\,q(p)-1}\right] = \frac{1}{i\,q(p)-1}\left[\frac{1}{i\,q(p)-1}\right],$

$= \frac{1}{(i\,q(p)-1)^2} \cdot \frac{(-i\,q(p)-1)^2}{(-i\,q(p)-1)^2} = \frac{1-(q(p))^2}{((q(p))^2+1)^2} + \frac{2\,q(p)}{((q(p))^2+1)^2} i.$

$T_g^c\big(sinh\,(a\,lnx)\big) = \frac{-a\left[\left(q(p)\right)^2-1\right)+a^2}{(q(p))^2+1)^2+2a^2(q(p))^2+a^4} + \frac{2a\,q(p)}{(q(p))^2+1)^2+2a^2(q(p))^2-1)+a^4} i$, where a is a constant, $i = \sqrt{-1}\,, |q(p)-1| > a \;\; and \;\; x > 0.$

Proof

$T_g^c\;\big(sinh(a\,lnx)\big) = T_g^c\left(\frac{e^{alnx}-e^{-alnx}}{2}\right) = \frac{1}{2} T_g^c(x^a - x^{-a})\;,$

$= \frac{1}{2}[T_g^c\;(x^a)\text{-}T_g^c\;(x^{-a})] = \frac{1}{2}\left[\frac{1}{i\,q(p)-(a+1)} - \frac{1}{i\,q(p)—a+1}\right],$

$= \frac{1}{2}\left[\frac{i\,q(p)+a-1-iq(p)+a+1}{(i\,q(p)-a-1)(i\,q(p)+a-1)}\right] = \frac{2a}{2\big((i\,q(p)-1)-a\big)\big((i\,q(p)-1)+a\big)}\;,$

$= \frac{a}{(i\,q(p)-1)^2-a^2} \cdot \frac{(-i\,q(p)-1)^2-(a)^2}{(-i\,q(p)-1)^2-(a)^2}\;,$

$= \frac{a\left[(-i\,q(p)-1)^2-a^2\right]}{\left(\left(q(p)\right)^2+1\right)^2+2a^2((q(p))^2-1)+a^4}\;,$

$= \frac{a\Big(\big(-(q(p))^2+1-a^2\big)+2i\,q(p)\Big)}{\left(\left(q(p)\right)^2+1\right)^2+2a^2\,(q\,(p))^2-1)+a^4}\;,$

$= \frac{a-\big(a(q(p))^2+a^3\big)}{\left(\left(q(p)\right)^2+1\right)^2+2a^2\,((q(p))^2-1)+a^4} + \frac{2ap}{\left(\left(q(p)\right)^2+1\right)^2+2a^2\,((q(p))^2-1)+a^4} i\;,$

$= \frac{a-\left[((q(p))^2-1)+a^2\right]}{((q(p))^2+1)^2+2a^2\,((q(p))^2-1)+a^4} + \frac{2ap}{((q(p))^2+1)^2+2a^2\,((q(p))^2-1)+a^4} i.$

$T_g^c\;\big(cosh\;(a\;lnx\;)\big) = \frac{-\,(\left(q(p)\right)^2+1)+a^2}{((q(p))^2+1)^2+2\,a^2((q(p))^2-1\,)+a^4} - \frac{q(p)\Big(\left(q(p)\right)^2+1\Big)+a^2)}{((q(p))^2+1)^2+2\,a^2((q(p))^2-1\,)+a^4} i$, where a is a constant, $i = \sqrt{-1}$, $|q(p)-1| > a \; and \; x > 0.$

Proof

$T_g^c\big(cosh\;(a\;ln\;x)\big) = T_g^c\left(\frac{e^{alnx}+e^{-alnx}}{2}\right) = \frac{1}{2} T_g^c(x^a + x^{-a})\;,$

$= \frac{1}{2}\;\;[T_g^c\;(x^a) + T_g^c\;(x^{-a})],$

$= \frac{1}{2}\left[\frac{1}{i\,q(p)-(a+1)} + \frac{1}{i\,q(p)—(-a+1)}\right] = \frac{1}{2}\left[\frac{i\,q(p)+a-1+iq(p)-a-1}{(i\,q(p)-1-a)(iq(p)-1+a)}\right],$

$$= \frac{1}{2}\left[\frac{2i\,q(p)-2}{(iq(p)-1)^2-a^2}\right] = \frac{i\,q(p)-1}{(i\,q(p)-1)^2-a^2} \cdot \frac{(-i\,q(p)-1)^2-a^2}{(-i\,q(p)-1)^2-a^2}\ ,$$

$$= \frac{(i\,q(p)-1)((-i\,q(p)-1)^2-a^2)}{\left((q(p))^2+1\right)^2+2a^2((q(p))^2-1)+a^4}\ ,$$

$$= \frac{(-(q(p))^2+1-a^2+2i\,q(p))(i\,q(p)-1)}{\left((q(p))^2+1\right)^2+2a^2((q(p))^2-1)+a^4}\ ,$$

$$= \frac{-i(q(p))^3+i\,q(p)-i\,q(p)a^2-2(q(p))^2+(q(p))^2-1+a^2-2i\,q(p)}{\left((q(p))^2+1\right)^2+2a^2((q(p))^2-1)+a^4}\ ,$$

$$= \frac{-(q(p))^2+a^2-1}{\left((q(p))^2+1\right)^2+2a^2((q(p))^2-1)+a^4} - \frac{[(q(p))^3+a^2\,q(p)+q(p)]}{\left((q(p))^2+1\right)^2+2a^2((q(p))^2-1)+a^4}\,i\ ,$$

$$= \frac{-((q(p)^2+1)+a^2}{((q(p))^2+1)^2+2a^2((q(p))^2-1)+a^4} - \frac{q(p)((q(p)^2+1)+a^2)}{\left((q(p))^2+1\right)^2+2a^2((q(p))^2-1)+a^4}\,i\ .$$

$T_g^c\ \left(sin(a\,lnx\,)\right) = \frac{-a[(q(p))^2-1)-a^2]}{((q(p))^2+1)^2-2a^2(q(p))^2-1)+a^4} + \frac{2a\,q(p)}{((q(p))^2+1)^2-2a^2(q(p))^2-1)+a^4}\,i,$ where a is a constant, $i = \sqrt{-1}$, $x > 1$ *and* $q(p) > 1$.

Proof

$$T_g^c\ (\sin(a\,lnx)) = T_g^c\left(\frac{e^{ialnx}-e^{-ialnx}}{2i}\right) = \frac{1}{2i}T_g^c(x^{ia}-x^{-ia}),$$

$$= \frac{1}{2i}\left[\frac{1}{i\,q(p)-(ia+1)} - \frac{1}{i\,q(p)-(-ia+1)}\right],$$

$$= \frac{1}{2i}\left[\frac{i\,q(p)+ia-1-iq(p)+ia+1}{(i\,q(p)-ia-1)(i\,q(p)+ia-1)}\right],$$

$$= \frac{2ia}{2i((i\,q(p)-1)-ia)((i\,q(p)-1)+ia)}\ ,$$

$$= \frac{a}{(i\,q(p)-1)^2-a^2} \cdot \frac{(-i\,q(p)-1)^2+a^2}{(-i\,q(p)-1)^2+a^2}\ ,$$

$$= \frac{a[(-iq(p)-1)^2+a^2]}{((q(p))^2+1)^2-2a^2((q(p))^2-1)+a^4}\ ,$$

$$= \frac{a\left((-(q(p))^2+1+a^2)+2ip\right)}{\left((q(p))^2+1\right)^2-2a^2((q(p))^2-1)+a^4}\ ,$$

$$= \frac{a+a^3-a(q(p))^2}{\left((q(p))^2+1\right)^2-2a^2((q(p))^2-1)+a^4} + \frac{2a\,q(p)}{\left((q(p))^2+1\right)^2-2a^2((q(p))^2-1)+a^4}\,i\ ,$$

$$= \frac{-a[((q(p))^2-1)-a^2]}{\left((q(p))^2+1\right)^2-2a^2((q(p))^2-1)+a^4} + \frac{2a\,q(p)}{\left((q(p))^2+1\right)^2-2a^2((q(p))^2-1)+a^4}\,i.$$

$T_g^c\left(cos\ (a\ lnx)\right) = \frac{-a\left[(q(p))^2[+1)+a^2\right]}{((q(p))^2+1)^2-2a^2(q(p))^2-1)+a^4} - \frac{q(p)((q(p)^2+1)-a^2)}{((q(p))^2+1)^2-2a^2(q(p))^2-1)+a^4}\,i,$ where a is a constant, $i = \sqrt{-1}$, $x > 1$ and $q(p) > 1$.

Proof

$$T_g^c\ (\cos(a\,ln\,x)) = T_g^c\left(\frac{e^{ialnx}+e^{-ialnx}}{2}\right),$$

$$= \frac{1}{2}T_g^c(x^{ia}+x^{-ia}),$$

$$= \frac{1}{2}\left[\frac{1}{i\,q(p)-(ia+1)} + \frac{1}{iq(p)-(-ia+1)}\right],$$

$$= \frac{1}{2}\left[\frac{iq(p)+ia-1+iq(p)-ia-1}{(i\,q(p)-1-ia)(ip-1+ia)}\right],$$

$$= \frac{1}{2}\left[\frac{2iq(p)-2}{(iq(p)-1)^2+a^2}\right]\ ,$$

$$= \frac{iq(p)-1}{(iq(p)-1)^2+a^2} \cdot \frac{(-iq(p)-1)^2+a^2}{(-iq(p)-1)^2+a^2}\ ,$$

$$= \frac{(iq(p)-1)((-iq(p)-1)^2+a^2)}{((q(p))^2+1)^2-2a^2((q(p))^2-1)+a^4},$$

$$= \frac{(-(q(p))^2+1+a^2+2iq(p))(iq(p)-1)}{((q(p))^2+1)^2-2a^2\left((q(p))^2-1\right)+a^4},$$

$$= \frac{-i(q(p))^3+iq(p)+iq(p)a^2-2(q(p))^2+(q(p))^2-1-a^2-2iq(p)}{((q(p))^2+1)^2-2a^2\left((q(p))^2-1\right)+a^4},$$

$$= \frac{-((q(p))^2+a^2+1)}{((q(p))^2+1)^2-2a^2\left((q(p))^2-1\right)+a^4} + \frac{(-(q(p))^3+a^2q(p)-q(p))}{((q(p))^2+1)^2-2a^2\left((q(p))^2-1\right)+a^4}\,i\ ,$$

$$= \frac{-[(q(p))^2+1)+a^2]}{((q(p))^2+1)^2-2a^2(q(p))^2-1)+a^4} - \frac{q(p)((q(p)^2+1)-a^2)}{((q(p))^2+1)^2-2a^2(q(p))^2-1)+a^4}\,i.$$

T_g^c $(x^m\ f(x))$, where $f(x) = sin\ (a\ lnx)\ or\ f(x) = cos(a\ lnx), m \in \mathbb{R}$, $x > 0$ and a is a constant.

$$T_g^c(x^m\ f(x)) = \frac{-a[(q(p))^2-(m+1)^2-a^2]}{((q(p))^2+(m+1)^2)^2-2a^2(q(p))^2-(m+1)^2)+a^4} + \frac{2a(m+1)q(p)}{((q(p))^2+(m+1)^2)^2-2a^2(q(p))^2-(m+1)^2)+a^4}\,i\ .$$

Proof

$$T_g^c(x^m\sin(alnx)) = T_g^c\left(\frac{x^m e^{ialnx} - x^m e^{-ialnx}}{2i}\right),$$

$$= \frac{1}{2i}T_g^c(x^{ia+m} - x^{-ia+m}),$$

$$= \frac{1}{2i}\ [T_g^c\ (x^{ia+m}) - T_g^c\ (x^{-ia+m})],$$

$$= \frac{1}{2i}\left[\frac{1}{i\,q(p)-(ia+m+1)} - \frac{1}{iq(p)-(-ia+m+1)}\right],$$

$$= \frac{1}{2i}\left[\frac{iq(p)+ia-1-m-iq(p)+ia+1+m}{(i\,q(p)-ia-(m+1))(iq(p)+ia)-(m+1))}\right],$$

$$= \frac{a}{(i\,q(p)-(m+1))^2+a^2}\cdot\frac{(-i\,q(p)-(m+1))^2+a^2}{(-iq(p)-(m+1))^2+a^2},$$

$$= \frac{a[-iq(p)-(m+1)^2+a^2]}{((q(p))^2+(m+1)^2)^2-2a^2((q(p))^2-(m+1)^2)+a^4},$$

$$= \frac{a((-q(p))^2+(m+1)+a^2)+2\,(m+1)i\,q(p)}{(q(p))^2+(m+1)^2)^2-2a^2((q(p))^2-(m+1)^2)+a^4},$$

$$= \frac{a(m+1)^2+a^3-a(q(p))^2}{(q(p))^2+(m+1)^2)^2-2a^2((q(p))^2-(m+1)^2)+a^4} + \frac{2(m+1)\,aq(p)}{(q(p))^2+(m+1)^2)^2-2a^2((q(p))^2-(m+1)^2)+a^4}\,i\ ,$$

$$= \frac{-a[(q(p))^2-(m+1)^2)-a^2]}{(q(p))^2+(m+1)^2)^2-2a^2((q(p))^2-(m+1)^2)+a^4} + \frac{2a(m+1)\,q(p)}{(q(p))^2+(m+1)^2)^2-2a^2((q(p))^2-(m+1)^2)+a^4}\,i.$$

By the same way, it is possible to prove:

$$T_g^c\ (x^m\cos(alnx)) = \frac{-(m+1)[(q(p))^2+(m+1)^2)+a^2]}{((q(p))^2+(m+1)^2)^2-2a^2((q(p))^2-(m+1)^2)+a^4} - \frac{q(p)((q(p))^2+(m+1)^2)-a^2)}{(q(p))^2+(m+1)^2)^2-2a^2((q(p))^2-(m+1)^2)+a^2}\,i\ .$$

The Generalization Of Inverse Complex Al-Tememe Integral Transform For Fundamental Functions

(1) $T_g^{c-1}\left(\frac{-1}{1+(q(p))^2} - \frac{q(p)}{1+(q(p))^2}\,i\right) = 1$.

(2) $T_g^{c-1}\left(\frac{-(n+1)}{(q(p))^2+(n+1)^2} - \frac{q(p)}{(q(p))^2+(n+1)^2}\,i\right) = x^n, n \in \mathbb{R}, i = \sqrt{-1}.$

(3) $T_g^{c-1}\left(\frac{1-(q(p))^2}{((q(p))^2+1)^2}+\frac{2\,q(p)}{((q(p))^2+1)^2}i\right)=\ln x\ ,x>1\ ,i=\sqrt{-1}.$

(4) $T_g^{c-1}\left(\frac{-a\left[\left(q(p)\right)^2-1\right)+a^2]}{(q(p))^2+1)^2+2a^2(q(p))^2-1)+a^4}+\frac{2a\,q(p)}{(q(p))^2+1)^2+2a^2(q(p))^2-1)+a^4}i\right)=\sinh(a\,lnx),$ a is a constant i $=\sqrt{-1}$ $and\ x>1$.

(5) $T_g^{c-1}\left(\frac{-\left(\left(q(p)\right)^2+1\right)+a^2}{(q(p))^2+1)^2+2a^2(q(p))^2-1)+a^4}-\frac{q(p)((q(p))^2+1)+a^2}{(q(p))^2+1)^2+2a^2(q(p))^2-1)+a^4}i\right)=\cosh(a\,lnx),$ a is a constant i $=\sqrt{-1}$ $and\ x>1$.

(6) $T_g^{c-1}\left(\frac{-a\left[\left(q(p)\right)^2-1\right)+a^2]}{(q(p))^2+1)^2-2a^2(q(p))^2-1)+a^4}+\frac{2a\,q(p)}{(q(p))^2+1)^2-2a^2(q(p))^2-1)+a^4}i\right)=\sin(a\,lnx),$ a is a constant , i $=\sqrt{-1}$ $and\ x>1$.

(7) $T_g^{c-1}\left(\frac{-[((q(p))^2+1)+a^2]}{(q(p))^2+1)^2-2a^2(q(p))^2-1)+a^4}-\frac{(q(p)((q(p))^2+1)+a^2}{(q(p))^2+1)^2-2a^2(q(p))^2-1)+a^4}i\right)=\cos(a\,lnx),$ a is a constant and $x>1$, i $=\sqrt{-1}$.

The Generalization Of Complex Al-Tememe Integral Transform For Derivatives

Theorem (I)

Let $y(x)$ be a function for the variable x, where $x>1$ and its derivatives $y'(x),y''\,(x),\dots,y^{(n)}(x)$ exist, then:

$T_g^c[x^n y^n\,(x)]=-y^{(n-1)}\,(1)-(iq(p)-n)y^{(n-2)}(1)-\dots-(iq(p)-n)(iq(p)\ -\ (n-1))\dots(iq(p)-2)y(1)+(iq(p)-n)\,(iq(p)-\ (n-1))\ (iq(p)-1)F(iq(p)),$ where n is a positive integer number.

Proof: Consider the following:

If $n=1$, then:

$T_g^c(xy')=-y(1)+(iq(p)-1)T_g^c\,(y)$,

$T_g^c(xy')=\int_1^\infty x^{-iq(p)}\mathrm{x}y'd\,x\ =\int_1^\infty x^{-iq(p)+1}\,y'd\,x$,

Integration by parts, gives:

$T_g^c\ (xy')=-y(1)+(iq(p)-1)T_g^c\,(y(x)).$

If $n\ =2$, then:

$T_g^c(x^2y'')=\ -y'(1)-(iq(p)-2)y(1)+\ (iq(p)-2)(iq(p)-1)T_g^c(y(x)),$

$T_g^c(x^2y'')=\int_1^\infty x^{-iq(p)}\ x^2y''\mathrm{d}\,x=\int_1^\infty x^{-iq(p)+2}\,y''dx,$

Also, performing integration by parts:

$=x^{-iq(p)+2}y'\,|\,_1^\infty\text{-}\int_1^\infty(-iq(p)+2)x^{-iq(p)+1}\,y'dx.$

If n=3, then:

$T_g^c(x^3y''')=-y''(1)-(iq(p)-3)\mathrm{y}'\ (1)-\ (iq(p)-3)(iq(p)\ \ y(1)+(iq(p)-3)(iq(p)-2)(iq(p)-1)T_g^c\,(y(x)$,

$T_g^c(x^3y''')=\int_1^\infty x^{-iq(p)}\ x^3y'''\mathrm{dx}=\int_1^\infty x^{-iq(p)+3}\,y'''dx$

$=x^{-iq(p)+3}y''\,|\,_1^\infty\text{-}\int_1^\infty(-iq(p)+3)x^{-iq(p)+2}\,y''dx$

$=-\mathrm{y}''\ (1)+(iq(p)-3)(-\mathrm{y}'\ (1)-(iq(p)-2)y(1)+(iq(p)-2)(iq(p)-1)T_g^c(y)),$

$= -y''(1) - (iq(p)-3)y'(1) - (yq(p)-3)(iq(p)-2)y(1) + (iq(p)-3)(iq(p)-2)(iq(p)-1)\, T_g^c(y))$.

In general:

$T_g^c(x^n y^{(n)}) = -y^{n-1}(1) - (iq(p)-n)y^{n-2}(1) - (iq(p)-n)(iq(p)-(n-1)y^{n-3}(1) - (iq(p)-n)\,(iq(p)-(n-1)) \dots (iq(p)-2)y(1) + (iq(p)-n)\,(iq(p)-(n-1)) \dots (iq(p)-2)(iq(p)-1)T_g^c(y)$, where n is a positive integer.

Implementing the Generalization of Complex Al-Tememe Transform Derivatives Theorem into Some Functions

Theorem (6.1) is going to be applied to the following functions as follows:

$T_g^c(lnx)^n = \frac{n!(iq(p)+1)^{n+1}}{((q(p))^2+1)^{n+1}}$, if n is an odd number, and
$T_g^c(lnx)^n = \frac{-(n!)(iq(p)+1)^{n+1}}{((q(p))^2+1)^{n+1}}$, if n is an even number.
If $n = 1$, then:
$T_g^c(lnx) = \frac{(iq(p)+1)^2}{(q(p))^2+1)^2}$.

If $n = 2$, then:
Let $y = (lnx)^2$, with conditions: $y(1) = 0$, $y' = 2\ln x .\frac{1}{x}$ that is $xy' = 2\ln x$.
$T_g^c(xy')= 2T_g^c(\ln x)$ that is: $-y(1) + (iq(p)-1)T_g^c(y) = \frac{2}{(iq(p)-1)^2}$.
$\therefore T_g^c(y) = \frac{-(i\square(p)+1)^3}{(q(p))^2+1)^3}$.

If $n = 3$, then:

Let, $y = (lnx)^3$, with condition: $y(1) = 0$,
$y' = 3(lnx)^2.\frac{1}{x}$ that is: $xy' = 3(lnx)^2$
$T_g^c(xy') = 3T_g^c(lnx)^2$
$-y(1) + (iq(p)-1)\, T_g^c(y) = \frac{3.2}{(iq(p)-1)^3}$,
$T_g^c(y) = \frac{3!(iq(p)+1)^4}{(q(p))^2+1)^4}$.
And for $y = (lnx)^n$, with condition: $y(1) = 0$.
$y' = n(lnx)^{n-1}.\frac{1}{x}$ that is: $xy' = n(lnx)^{n-1}$
$T_g^c(xy') = nT_g^c(lnx)^{n-1}$
$-y(1) + (iq(p)-1)\, T_g^c(y) = \frac{n.(n-1)!}{(iq(p)-1)^n}$,
$T_g^c(y) = \frac{n!(-iq(p)-1)^{n+1}}{(q(p))^2+1)^{n+1}}$
$= \frac{n!(iq(p)-1)^{n+1}}{(q(p))^2+1)^{n+1}}$, when n is odd.
$= \frac{-(n!)(iq(p)+1)^{n+1}}{(q(p))^2+1)^{n+1}}$ when n is even.

Theorem (I) is going to be applied to find: $T_g^c(x^m(lnx)^n)$, for different positive integer values of m and n.

First case: if n=1, then:

$$T_g^c(x^m(\ln x)) = \frac{1}{(iq(p)-(m+1))^2}.\frac{(-iq(p)-(m+1))^2}{(-iq(p)-(m+1))^2} = \frac{(iq(p)+(m+1))^2}{(iq(p))^2+(m+1)^2)^2}\ ;\ m \in N,\ x > 0.$$

Second case: if n=2 then: $T_g^c(x^m(\ln x)^2)$ multiple values of m are going to be discussed:

If m=1, then: $T_g^c(x\,(\ln x)^2)$

Let $y\ = x\,(lnx)^2$, with the condition $y(1) = 0$,

$y' = x.2lnx.\frac{1}{x} + (lnx)^2$,

$xy' = 2xlnx\ +\ x(lnx)^2$,

$T_g^c(xy') = 2T_g^c(x\ lnx) +\ T_g^c(y)$,

$-y(1) + (iq(p) - 1)\ T_g^c(\mathrm{y}) = \frac{2}{(iq(p)-2)^2} + T_g^c(y)$,

$(iq(p) - 1)\ T_g^c(y)\ - T_g^c(y)\ = \frac{2}{(iq(p)-2)^2}$,

$T_g^c(y) = \frac{2}{(iq(p)-2)^3}.\frac{(-iq(p)-2)^3}{(-iq(p)-2)^3} = \frac{-2(iq(p)+2)^3}{((\mathrm{q(p)})^2+4)^3}$.

If m =2, then: $T_g^c(x^2\,(\ln x)^2)$

Let y=$x^2\,(lnx)^2$, with the condition $y(1) = 0$,

$\mathrm{y}' = x^2.2ln\,x.\frac{1}{x} + 2x\,(lnx)^2$,

$x\,y' = 2x^2\ ln\,x\ + 2x^2\,(lnx)^2$,

$T_g^c(x\,y')\ = 2T_g^c(x^2\,(lnx)\ + 2T_g^c(y))$,

$-y(1) + (iq(p) - 1)\ T_g^c(y)\ = \frac{2}{(iq(p)-3)^2} + 2T_g^c(y)$,

$(iq(p) - 1)\ T_g^c(y)\ - 2T_g^c(y)\ = \frac{2}{(iq(p)-3)^2}$,

$T_g^c(y) = \frac{2}{(iq(p)-3)^3}.\frac{(-iq(p)-3)^3}{(-iq(p)-3)^3} = \frac{-2(iq(p)+3)^3}{(iq(p))^2)+9)^3}$.

If m=3, then: $T_g^c(x^3\,(lnx)^2)$

Let $y = (x^3\,(\ln x)^2)$, with the condition $y(1) = 0$,

$y'\ = x^3.2ln\,x.\frac{1}{x} + 3x^2\,(\ln x)^2$,

$x\,y' = 2x^3\ ln\,x\ + 3x^3\,(lnx)^2$,

$T_g^c(x\,y')\ = 2T_g^c(x^3\,(lnx)\ + 3T_g^c(y)$,

$-y(1) + (iq(p) - 1)\ T_g^c(\mathrm{y}) = \frac{2}{(iq(p)-4)^2} + 3T_g^c(y)$,

$(iq(p) - 1)\ T_g^c(\mathrm{y}) - 3T_g^c(\mathrm{y}) = \frac{2}{(iq(p)-4)^2} = \frac{-2(iq(p)+4)^3}{(iq(p))^2)+16)^3}$.

In general, $T_g^c(x^m\,(lnx)^2)$, $m \in N$, $x > 0$

Let $y = x^m\,(\ln x)^2$, with the condition $y(1) = 0$,

$y'\ = x^m.2ln\,x.\frac{1}{x} + mx^{m-1}\,(lnx)^2$,

$x\,y'\ = 2x^m\ ln\,x + mx^m\,(lnx)^2$,

$T_g^c(x\,y')\ = 2T_g^c(x^m\,(lnx)\ + mT_g^c(y)$,

$-y(1) + (iq(p) - 1)\ T_g^c(\mathrm{y}) = \frac{2}{(iq(p)-(\mathrm{m}+1))^2} + mT_g^c(y)$,

$(iq(p) - 1)\ T_g^c(\mathrm{y}) - mT_g^c(\mathrm{y}) = \frac{2}{(iq(p)-(m+1))^2}$,

$$T_g^c(y) = \frac{2}{(iq(p)-(\mathrm{m}+1))^3}.\frac{(-iq(p)-(\mathrm{m}+1))^3}{(-iq(p)-(\mathrm{m}+1))^3} = \frac{-2(iq(p)+(\mathrm{m}+1))^3}{((q(p))^2+(\mathrm{m}+1))^3}, m \in N,\ i = \sqrt{-1}\ .$$

Third case: To find: $T_g^c(x^m(lnx)^3)$ multiple values of m are going to be discussed:

If m=1, then: $T_g^c(x\,(lnx)^3)$
Let $y = x\,(lnx)^3$, with the condition $y(1) = 0$,
$x\,y' = 3\,x\,(lnx\,)^2 + x\,(lnx)^3$,
$T_g^c\,x\,(y') = 3T_g^c\,(lnx)^2) + T_g^c(y)$,
$-y(1) + (iq(p) - 1)\,T_g^c(y) = 3.\frac{2}{(iq(p)-2)^3} + T_g^c(y)$,
$(iq(p) - 1)\,T_g^c(y) - T_g^c(y) = \frac{3!}{(iq(p)-2)^3}$,
$T_g^c(y) = \frac{3!}{(iq(p)-2)^4}.\frac{(-iq(p)-2)^4}{(-iq(p)-2)^4} = \frac{3!(iq(p)+2)^4}{((q(p))^2)+4)^4}$.

If m= 2, then: $T_g^c(x^2(lnx)^3)$
Let $y = x^2\,(lnx)^3$, with the condition $y(1) = 0$,
$xy' = 3x^2\,(lnx\,)^2 + 2x^2(lnx)^3$,
$T_g^c(xy') = 3T_g^c(x^2(lnx)^2) + 2\,T_g^c(y)$,
$(iq(p) - 1)\,T_g^c(y) - 2T_g^c(y) = \frac{3!}{(iq(p)-3)^3}$,
$T_g^c(y) = \frac{3!}{(iq(p)-3)^4}.\frac{(-iq(p)-3)^4}{(-iq(p)-3)^4} = \frac{3!(iq(p)+3)^4}{(iq(p))^2)+9)^4}$.

If m= 2, then: $T_g^c(x^3(lnx)^3)$
Let $y = x^3\,(lnx)^3$, with the condition $y(1) = 0$,
$\mathrm{y}' = x^3.3(lnx)^2.\frac{1}{x} + 2x\,(lnx)^3$,
$x\,\mathrm{y}' = 3x^2\,(ln\,x\,)^2 + 3x^2(lnx)^3$,
$T_g^c(xy') = 3T_g^c(x^3(lnx)^2) + 3\,T_g^c(y)$,
$(iq(p) - 1)\,T_g^c(\mathrm{y}) - 3T_g^c(\mathrm{y}) = \frac{3!}{(iq(p)-4)^3}$,
$T_g^c(\mathrm{y}) = \frac{3!}{(iq(p)-4)^4}.\frac{(-iq(p)-4)^4}{(-iq(p)-4)^4} = \frac{3!(-iq(p)+4)^4}{(iq(p))^2)+16)^4}$.

In general: $(T_g^c(x^m(lnx)^3),\ m \in N$ and $x > 0$
Let $y = x^m\,(\ln x)^3$, with the condition $y(1) = 0$,
$\mathrm{y}' = x^m.3(lnx)^2.\frac{1}{x} + mx^{m-1}\,(lnx)^3$,
$x\mathrm{y}' = 3x^m\,(lnx\,)^2 + mx^m(lnx)^3$,
$T_g^c(x\mathrm{y}') = 3T_g^c(x^m(lnx)^2) + m\,T_g^c(y)$,
$(iq(p) - 1)\,T_g^c(y) - mT_g^c(\mathrm{y}) = \frac{3!}{(iq(p)-(m+1))^3}$,
$T_g^c(y) = \frac{3!}{(iq(p)-(\mathrm{m}+1))^4}.\frac{(-iq(p)-(\mathrm{m}+1))^4}{(-iq(p)-(\mathrm{m}+1))^4} = \frac{3!(-iq(p)+(\mathrm{m}+1))^4}{(q(\mathrm{p}))^2+(m+1)^2)^4}$.

General case: Finding $T_g^c(x^m(\ln x)^n),\ m \in \mathbb{R}\ ,n \in N\,,x > 0$
Let $y = x^m\,(lnx)^n$, with the condition $y(1) = 0$,
$\mathrm{y}' = x^m.n(lnx)^{n-1}.\frac{1}{x} + mx^{m-1}\,(ln)x^n$,
$x\mathrm{y}' = nx^m\,(\ln x\,)^{n-1} + mx^m(lnx)^n$,
$T_g^c(x\mathrm{y}') = nT_g^c(x^m(lnx)^{n-1}) + m\,T_g^c(\mathrm{y})$,

$-y(1) + (iq(p) - 1)\ T_g^c(\mathrm{y}) = n.\frac{(\mathrm{n}-1)!}{(iq(p)-(\mathrm{m}+1))^{\mathrm{n}}} + mT_g^c(y),$

$(iq(p) - 1)\ T_g^c(\mathrm{y}) - mT_g^c(y) = \frac{n!}{(iq(p)-(m+1))^{n}}$,

$T_g^c(\mathrm{y}) = \frac{\mathrm{n}!}{(iq(p)-(\mathrm{m}+1))^{\mathrm{n}+1}} . \frac{(-iq(p)-(\mathrm{m}+1))^{\mathrm{n}+1}}{(-iq(p)-(\mathrm{m}+1))^{\mathrm{n}+1}} = \frac{\mathrm{n}!(-iq(p)-(\mathrm{m}+1))^{\mathrm{n}+1}}{(iq(p))^2)+(\mathrm{m}+1)^2)^{\mathrm{n}+1}}$, $m \in \mathbb{R}\ , n \in N.$

Differentiation And Integration Of The Generalization Of Complex Al-Tememe Integral Transform

Theorem (II): Differentiation of the Generalization of Complex Al-Tememe Integral Transform

For a piecewise and continuous function $f(x)$, through the period $[1,\infty)$, and $T_g^c(f(x)) = F(i\,q(p))$, $q(p)$ is a function of the p integral parameter, then:

$$\frac{d^k}{dq(p)^k} T_g^c\ (f(x)) = (-i)^k\ T_g^c[(ln(x))^k\ f(x)],\ k = 1,2, \dots$$

Proof: to prove this theorem the generalization of Complex Al-Tememe transform definition is taken: $T_g^c(f(x)) = \int_1^\infty x^{-iq(p)}\, f(x)dx.$

If $k = 1$, then: $\frac{d}{dq(p)}\ T_g^c\ (f(x))$

$= \frac{d}{dq(p)} \int_1^\infty\ x^{-iq(p)}\ f(x)dx,$

$= \int_1^\infty \frac{d}{dq(p)}\ x^{-iq(p)} f(x)dx,$

Since, $x^{-iq(p)} = e^{-iq(p)ln(x)}$ and, $\frac{d}{dq(p)}(x^{-iq(p)}) = -ix^{-iq(p)}ln(x),$

Then, $\frac{d}{dq(p)}\ T_g^c\ (f(x)) = \int_1^\infty -i\, x^{-iq(p)}\ ln(x))f(x)dx,$

$= (-i)^1 \int_1^\infty \ln(x)\, x^{-iq(p)}\, f(x)dx,$

$\therefore \frac{d}{dq(p)}\ T_g^c\ (f(x)) = (-i)^1\ T_g^c\ [ln(x)\, f(x)].$

If k=2, then: $\frac{d^2}{d(q(p))^2}\ T_g^c\big(f(x)\big)$

$= \frac{d}{dq(p)}[\frac{d}{dq(p)} T_g^c(f(x))],$

$= \frac{d}{dq(p)}[\ (-i)^1\ T_g^c\ [ln(x)\, f(x)]\ ],$

$= (-i)^1 \frac{d}{dq(p)} \int_1^\infty \ln(x) x^{-iq(p)}\, f(x)\ dx,$

$= (-i)^1 \int_1^\infty\ \frac{d}{dq(p)} x^{-iq(p)}\ \ln(x)\ f(x)\ dx,$

$= (-i)^1 \int_1^\infty\ (-i\, x^{-iq(p)} \ln x)\ lnx\, f(x)\ dx,$

$= (-i)^2 \int_1^\infty\ x^{-iq(p)}\, (\ln x)^2\ f(x)\ dx,$

$\therefore \frac{d^2}{d(q(p))^2}\ T_g^c\ (f(x)) = (-i)^2\ T_g^c\ [(lnx)^2\, f(x)].$

In General:

$\frac{d^k}{dq(p)^k} T_g^c\big(f(x)\big) = (-i)^{k-1}\ \int_1^\infty \frac{d}{d\,q(p)}\ x^{-iq(p)}\ (lnx)^{k-1}\, f(x)\ dx,$

$= (-i)^{k-1}\ \int_1^\infty \big(-i\, x^{-iq(p)}\ lnx\big)(lnx)^{k-1}\, f(x)\ dx,$

$= (-i)^k\ \int_1^\infty x^{-iq(p)} (lnx)^k\, f(x)\ dx,$

Now,
$\frac{d^k}{dq(p)^k} T_g^c[(f(x))] = (-i)^k \, T_g^c[\,(lnx)^k fx\,]$, where $k = 1,2,3 \ldots$
$\therefore T_g^c[(lnx)^k f(x)] = (i)^k \, \frac{d^k}{dq(p)^k} T_g^c[f(x)]$, where $k = 1,2,3 \ldots$

Theorem (III): Integration of the Generalization of Complex Al-Tememe Integral Transform

If $T_g^c[f(x)] = F(iq(p))$, then: $T_g^c\ (\int_1^x f(t)dt) = \frac{-1-iq(p)}{1+(q(p))^2} T_g^c\{x\, f(x)\}$.

Proof

Let $\int_1^x \mathrm{f(t)dt} = g(x)$, $\mathrm{g}'\ (x) = f(x)$, with the condition $g(1) = 0$,
$T_g^c\ (\int_1^x f(t)dt) = T_g^c\ (g(x)) \ = \ \int_1^\infty x^{-iq(p)}\, g(x)dx$,
Integration by parts, gives:
$= \frac{1+iq(p)}{1+(q(p))^2}\, x^{-iq(p)+1} g(x)\, \Big|_1^\infty \quad - \int_1^\infty \frac{1+iq(p)}{1+(q(p))^2}\, x^{-iq(p)+1}\ \mathrm{g}'(x)dx$,
$= \frac{-1-iq(p)}{1+(q(p))^2} \int_1^\infty x^{-iq(p)}\ xg'(x)dx$,
$T_g^c\ (g(x)) = \frac{-1-iq(p)}{1+(q(p))^2} T_g^c[xf(x)]$,
$\therefore g(x) = T_g^{c-1}\left[\frac{-1-\mathrm{iq(p)}}{1+(\mathrm{q(p)})^2} T_g^c[\mathrm{xf(x)}]\right]$.

Solving Linear Ordinary Differential Equations Using The Generalization Of Complex Al-Tememe Transform

For an (n)-order linear variable coefficients ordinary differential with the general form as:

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \ldots + a_{n-1} x y' + a_n y = f(x)\ , \qquad (1)$$

Where:

$f(x)$ is a function that is continuous,
$a_0, a_1, \ldots, a_n$ are arbitrary constants,
$y^{(n)}$ is the nth derivative of $y(x)$,
$y(1), y(2), \ldots, y(n-1)$ (1) should be specified.

It is possible to solve Equation (1) by applying the generalization of the Complex Al-Tememe transform to both sides of the equation, and it can be written as:

$$T_g^c(y) = \frac{h(iq(p))}{k(iq(p))},\ k(iq(p)) \neq 0\ , i = \sqrt{-1}\ \ , q(p) > 0\ , \qquad (2)$$

Where, k is a polynomial with well-known prime cofactors, h *and* k are polynomials of $iq(p)$, the order of the h polynomial is below the order of the k polynomial

Applying T_g^{c-1} to both sides of Equation (2), obtains:

$$y \ = \ T_g^{c-1} = [\frac{h(iq(p))}{k(i\square(p))}], \quad k(iq(p)) \neq 0 \quad , \quad i = \sqrt{-1} \quad , \quad q(p) > 0\ , \qquad (3)$$

The polynomial $h(iq(p))$ is needlessly defined, and it's categorized by nothing but its symbol. However, the polynomial $k\,(iq\,(p))$ includes the denominator of generalization of the Complex Al-Tememe transform for the function $f(x)$ multiplied by $(a_0(iq(p))^n + a_1(iq(p))^{n-1} + \dots + an)$.

Equation (3) represents the general solution of the ordinary differential equation (1), whose form is given by:

$$y = A_0\, K_0(x) + A_1\, K_1(x) + \dots + A_m\, K_m\,(x)\ . \tag{4}$$

where, $K_0\,, K_1\,, \dots, K_m$ are functions of x, and $A_0\,, A_1\,, \dots\,, A_m$ are arbitrary constants, their numbers are equal to the order of $k(iq(p))$.

To calculate the values of $A_0\,, A_1\,, \dots\,, A_m$ the initial conditions are substituted, one of these conditions is: $y(1)$, to get:

$$A_0\, K_0(x) + A_1\, K_1(x) + \dots + A_m\, K_m\,(x) = y(1), \tag{5}$$

Taking the derivatives to Equation (5) for (m) times to obtain:

$$A_0\, K_0{}'(1) + A_1\, K_1{}'(1) + \dots + A_m\, K_m{}'(1) = y'(1),$$
$$A_0\, K_0{}''(1) + A_1\, K_1{}''(1) + \dots + A_m\, K_m{}''(1) = y''(1),$$
$$A_0\, K_0{}^{(m)}(1) + A_1\, K_1{}^{(m)}(1) + \dots + A_m\, K_m{}^{(m)}(1) = y^{(m)}(1).$$

This linear algebra system (model) can be solved for $A_0, A_1, \dots A_m$, the solution of the above system represents the required solution to the ordinary differential equation (1).

Notable mentions:

$T_g^c(\mathrm{xy}') = (iq(p) - 1)\, T_g^c(y) + a$, where a is a constant.
$T_g^c(c\mathrm{x}^2\mathrm{y}'') = (iq(p) - 2)(iq(p) - 1)\, T_g^c(y) + h_1(iq(p))$, where the order of the polynomial $h_1(iq(p))$ is less than 2.
$T_g^c(\mathrm{x}^3\mathrm{y}''') = (iq(p) - 3)\,(iq(p) - 2)(iq(p) - 1)\, T_g^c(\mathrm{y}) + h_2(iq(p)$, where the order of the polynomial $h_2(iq(p))$ is less than 3.
$T_g^c\big(\mathrm{x}^{\mathrm{m}}\mathrm{y}^{(\mathrm{m})}\big) = (iq(p) - m)\,(iq(p) - m + 1)\, \dots (iq(p) - 1) T_g^c\,(y) + h_{m-1}\big(iq(p)\big)$; where the order of the polynomial $h_{m-1}\big(iq(p)\big)$ is less than m.

Applications

Application (I)

To solve the ordinary differential equation: $xy' - 2y = x^4$, with the condition $y(1) = 0$.

Taking a generalization of the Complex Al-Tememe transform to both sides of the above equation, obtains:

$$T_g^c\,(y) = \frac{1}{(iq(p)-3)(iq(p)-5)}\,,$$

The aforementioned equation's general solution, and after taking the inverse generalization of the Complex Al-Tememe transform, can be written as:

$$y = Ax^2 + Bx^4 \ , \qquad (6)$$

The inclusion of the constants A and B in the equation requires the availability of two sets of linear data.

From $y(1) = 0$, $A + B = 0$ is obtained.

The second equation can be found by substituting the provided initial data $y(1) = 0$ into the above ordinary differential equation to obtain:

$y'(1) - 2y(1) = 1$, that implies: $y'(1) = 1$.

Taking the derivative to Equation (6) and substituting $y'(1) = 1$, gives:

$2A + 4B = 1.$

After simple computations: $A = \frac{-1}{2}$, $B = \frac{1}{2}$.

$\therefore y(x) = \frac{-1}{2} x^2 + \frac{1}{2} x^4.$

The concluded result represents the required exact solution to the given ordinary differential equation.

Application (II)

To solve the ordinary differential equation: $x^2y''3xy' - 3y = x^{-4}\ln x$, with the condition $y'(1) = y(1) = 0$ and $x > 0$.

Taking T_g^c to the above equation:

$$T_g^c(y) = \frac{1}{(iq(p)-2)(iq(p)+2)(iq(p)+3)^2}, \qquad (7)$$

Taking T_g^{c-1} to both sides of Equation (7), it is possible to write the general exact solution as:

$$y = Ax + Bx^{-3} + Cx^{-4} + Dx^{-4}\ln x \ , \qquad (8)$$

Where A, B, C and D are arbitrary constants, and four linear equations are required to find their values.

The first equation can be found by substituting $y(1) = 0$ into Equation (8). The second equation can be found by deriving the general exact solution to equation (8) and substituting the initial data $y'(1) = 0$, but finding the third and fourth equations requires additional conditions, which are:

$y''(1) + 3y'(1) - 3y(1) = 0$ implies that $y''(1) = 0$,

$y''(1) + 2y''(1) + 3y''(1) + 3y'(1) - 3y'(1) = 1$ implies that $y'''(1) = 1$.

Substituting into y, y', y'', y''' respectively, provides:

$y(1) = y'(1) = y''(1) = 0 \, , y'''(1) = 1,$

$A + B + C = 0,$

$A - 3B - 4C + D = 0,$

$12B + 20C - 9D = 0,$

$-60B - 120C + 74D = 1,$

After simple computations: $A = \frac{1}{100}$ $, B = \frac{-1}{4}$ $, C = \frac{6}{25}$ $, D = \frac{1}{5}$.

$$y(x) = \frac{1}{100}x - \frac{1}{4}x^{-3} + \frac{6}{25}x^{-4} + \frac{1}{5}x^{-4}\ln x,\ x > 0.$$

The concluded result represents the required exact solution to the given ordinary differential equation.

Discussion and Conclusions

This work represents a detailed study of a new integral transform that generalizes the well-known complex Al-Tememe integral. The new integral transform has been proposed under the name "Generalization of the Complex Al-Tememe integral transform". The comprehensive study of the new transform, including its application to fundamental functions, its derivation and integration, and its use in solving ordinary differential equations, proved its efficiency and capability to contribute as an effective integral tool and be used in further applications in multiple scientific areas.

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