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New integral transform for solving some kinds of differential equations

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***Abstract*---**Our aim in this paper is to introduce new integral transform which we will call it "Battor – AlTememe" transform. This transform is useful for solving many types of differential equations (ordinary and partial) and we will introduce some definitions, concepts and identities. Integral transform is used to solve differential and integral equations. In 2008 [2], the researcher introduced AlTememe transform, which is given by the integral T(f(x))=∫_1^∞▒x^(-p) f(x)dx=F(p), For a function f(x) which is defined on an interval (1, ∞), x^(-p) is the kernel of this transform and p is positive constant such that the above integral is converge.

***Keywords*---**integral transform, solving, differential, equations.

Introduction

Integral transform is used to solve differential and integral equations. In 2008 [2], the researcher introduced AlTememe transform, which is given by the integral $T\big(f(x)\big) = \int_1^{\infty} x^{-p} f(x)dx = F(p)$, For a function $f(x)$ which is defined on an interval $(1, \infty)$, x^{-p} is the kernel of this transform and p is positive constant such that the above integral is converge. Many researchers [3-8] are used this transform to solve different types of differential and integral equations with applications.

Definition (1) [1]: Integral transform

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Let f be defined function for $x \in (a, b)$, the integral transform for f whose symbol $F(\lambda)$ is defined as $F(\lambda) = \int_a^b H(\lambda, x) f(x) dx$, where H is a function of the variables λ and x, is the kernel of the integral transform and (a, b) are real numbers or $\mp\infty$ such that the last integral converges.

Definition (2) [2]: Al-Tememe transform
Let f be a function defined in [1, ∞], Al-Tememe transform is defined by (A.T) $\big(f(x)\big) = \int_1^\infty x^{-p} f(x)dx$, where the integral is converge and p is a positive number.

Definition (3): Battor-AlTememe transform
Let f be a function defined in [1, ∞], Battor-AlTememe transform is defined by the integral $BA\big(f(x)\big) = \lambda \int_1^\infty x^{-\frac{1}{\lambda}} f(x)dx = F(\lambda)$; $\lambda > 0$ such that this integral is converge.

Property (1): Battor-AlTememe transform has linearity property, that is $BA\big(af(x) \mp bg(x)\big) = a\, BA\big(f(x)\big) \mp b\, BA\big(g(x)\big)$; a and b are constants, f and g are defined in [1, ∞].

Proof:
$BA\big(af(x) \mp bg(x)\big) = \int_1^\infty x^{-\frac{1}{\lambda}} \big(af(x) \mp bg(x)\big)dx$
$= \int_1^\infty ax^{-\frac{1}{\lambda}} f(x)dx \mp \int_1^\infty bx^{-\frac{1}{\lambda}} g(x)dx$
$= a\int_1^\infty x^{-\frac{1}{\lambda}} f(x)dx \mp b\int_1^\infty x^{-\frac{1}{\lambda}} g(x)dx$
$= a\, BA\big(f(x)\big) \mp b\, BA\big(g(x)\big)$

BA-Transform of fundamental functions

In this section, we will introduce BA-transform of fundamental functions, like constants functions, logarithms functions, hyperbolic functions and triangular functions and other functions.

1) $BA(1) = \frac{\lambda^2}{1-\lambda}$; $\lambda \in (0,1)$
   Proof: $BA(1) = \lambda \int_1^\infty x^{-\frac{1}{\lambda}} dx = \lambda \frac{x^{-\frac{1}{\lambda}+1}}{-\frac{1}{\lambda}+1}\Big]_1^\infty = \frac{\lambda^2}{1-\lambda}$
2) $BA(k) = \frac{k\lambda^2}{1-\lambda}$; k is constant, $\lambda \in (0,1)$
   Proof: from (1) and linearity property.
3) $BA(x^n) = \frac{\lambda^2}{1-(n+1)\lambda}$; $\lambda \in \left(0, \frac{1}{n+1}\right)$
   Proof: $BA(x^n) = \lambda \int_1^\infty x^{-\frac{1}{\lambda}+n} dx = \lambda \frac{x^{-\frac{1}{\lambda}+(n+1)}}{-\frac{1}{\lambda}+(n+1)}\Big]_1^\infty = \frac{\lambda^2}{1-(n+1)\lambda}$
4) $BA(\ln(x)) = \frac{\lambda^3}{(1-\lambda)^2}$; $\lambda \in (0,1)$
   Proof: $BA(\ln(x)) = \lambda \int_1^\infty x^{-\frac{1}{\lambda}} \ln(x)\, dx = \lambda \left(\frac{x^{-\frac{1}{\lambda}+1}}{-\frac{1}{\lambda}+1} \ln(x)\right)\Big]_1^\infty - \lambda \int_1^\infty \frac{x^{-\frac{1}{\lambda}}}{-\frac{1}{\lambda}+1} dx = -\lambda \frac{x^{-\frac{1}{\lambda}+1}}{\left(-\frac{1}{\lambda}+1\right)^2}\Big]_1^\infty = \frac{\lambda^3}{(1-\lambda)^2}$

5) $BA((\ln(x))^2) = \frac{2\lambda^4}{(1-\lambda)^3}$; $\lambda \in (0,1)$

Proof: $BA((\ln(x))^2) = \lambda \int_1^\infty x^{-\frac{1}{\lambda}} (\ln(x))^2 dx$

$= \lambda \left(\frac{x^{-\frac{1}{\lambda}+1}}{-\frac{1}{\lambda}+1} \ln(x) \right)]_1^\infty - 2\lambda \int_1^\infty \frac{x^{-\frac{1}{\lambda}}}{-\frac{1}{\lambda}+1} \ln(x)\, dx$

$= \frac{2\lambda}{1-\lambda} BA(\ln(x)) = \frac{2\lambda}{1-\lambda} \cdot \frac{\lambda^3}{(1-\lambda)^2} = \frac{2\lambda^4}{(1-\lambda)^3}$

6) $BA((\ln(x))^3) = \frac{6\lambda^5}{(1-\lambda)^4}$; $\lambda \in (0,1)$

Proof: $BA((\ln(x))^3) = \lambda \int_1^\infty x^{-\frac{1}{\lambda}} (\ln(x))^3 dx$

$= \lambda \left(\frac{x^{-\frac{1}{\lambda}+1}}{-\frac{1}{\lambda}+1} (\ln(x))^3 \right)]_1^\infty - 3\lambda \int_1^\infty \frac{x^{-\frac{1}{\lambda}}}{-\frac{1}{\lambda}+1} (\ln(x))^2\, dx = \frac{3\lambda}{1-\lambda} BA((\ln(x))^2) = \frac{6\lambda^5}{(1-\lambda)^4}$

7) $BA((\ln(x))^n) = \frac{n!\,\lambda^{n+2}}{(1-\lambda)^{n+1}}$; $\lambda \in (0,1)$

Proof: by induction

8) $BA(\cosh(a\ln(x))) = \frac{\lambda^2(1-\lambda)}{(1-\lambda)^2 - a^2\lambda^2}$

Proof: $BA(\cosh(a\ln(x))) = \lambda \int_1^\infty x^{-\frac{1}{\lambda}} \cosh(a\ln(x)) dx = \lambda \int_1^\infty x^{-\frac{1}{\lambda}} \frac{x^a + x^{-a}}{2} dx =$ $\frac{1}{2}(BA(x^a) + BA(x^{-a})) = \frac{1}{2}(\frac{\lambda^2}{1-(a+1)\lambda} + \frac{\lambda^2}{1-(-a+1)\lambda}) = \frac{\lambda^2(1-\lambda)}{(1-\lambda)^2 - a^2\lambda^2}$

9) $BA(\sinh(a\ln(x))) = \frac{a\,\lambda^3}{(1-\lambda)^2 - a^2\lambda^2}$

Proof: $BA(\sinh(a\ln(x))) = \lambda \int_1^\infty x^{-\frac{1}{\lambda}} \sinh(a\ln(x)) dx = \lambda \int_1^\infty x^{-\frac{1}{\lambda}} \frac{x^a - x^{-a}}{2} dx =$ $\frac{1}{2}\left(BA(x^a) - BA(x^{-a})\right) = \frac{a\,\lambda^3}{(1-\lambda)^2 - a^2\lambda^2}$

10) $BA(\cos(a\ln(x))) = \frac{\lambda^2(1-\lambda)}{(1-\lambda)^2 + a^2\lambda^2}$

Proof: $BA(\cos(a\ln(x))) = \lambda \int_1^\infty x^{-\frac{1}{\lambda}} \cos(a\ln(x))\, dx = \lambda \int_1^\infty x^{-\frac{1}{\lambda}} \frac{x^{ai} + x^{-ai}}{2} dx =$ $\frac{1}{2}\left(BA(x^{ai}) + BA(x^{-ai})\right) = \frac{\lambda^2(1-\lambda)}{(1-\lambda)^2 + a^2\lambda^2}$

11) $BA(\sin(a\ln(x))) = \frac{a\,\lambda^3}{(1-\lambda)^2 + a^2\lambda^2}$

Proof: $BA(\sin(a\ln(x))) = \lambda \int_1^\infty x^{-\frac{1}{\lambda}} \sinh(a\ln(x))\, dx = \lambda \int_1^\infty x^{-\frac{1}{\lambda}} \frac{x^a - x^{-a}}{2i} dx$

$= \frac{1}{2i}\left(BA(x^a) - BA(x^{-a})\right) = \frac{a\,\lambda^3}{(1-\lambda)^2 + a^2\lambda^2}$

Examples: In this section, we give some examples.

1. $BA(x) = \frac{\lambda^2}{1-2\lambda}$
2. $BA(x^{-1}) = \lambda^2$
3. $BA((\ln(x))^5) = \frac{5!\,\lambda^7}{(1-\lambda)^6}$
4. $BA(\cosh(2\ln(x))) = \frac{\lambda^2(1-\lambda)}{(1-\lambda)^2 - 4\lambda^2}$
5. $BA(\sinh(3\ln(x))) = \frac{3\,\lambda^3}{(1-\lambda)^2 - 9\lambda^2}$
6. $BA\left(\cos(\frac{1}{2}\ln(x))\right) = \frac{\lambda^2\,(1-\lambda)}{(1-\lambda)^2 + \frac{1}{4}\lambda^2}$

7. $BA\left(\sin(\frac{3}{2}\ln(x))\right) = \frac{\frac{3}{2}\lambda^3}{(1-\lambda)^2+\frac{9}{4}\lambda^2}$

Definition (4): Inverse of Battor-AlTememe transform:
If $BA(f(x)) = F(\lambda)$ is Battor-AlTememe transform, then we call $f(x) = (BA)^{-1}(F(\lambda))$ the inverse of Battor-AlTememe transform.

Property (2): the $(BA)^{-1}(F(\lambda))$ has linearity property, i.e
$(B.A)^{-1}(a\,F_1(\lambda) \mp b\,F_2(\lambda))$
$= a\left(BA)^{-1}(F_1(\lambda)) \mp b(BA)^{-1}\,(F_2(\lambda)\right) = af_1(x) \mp b\,f_2(x)$
Where a and b are constants.
Example (8): to find $(BA)^{-1}\frac{\lambda^3}{(1-2\lambda)\,(1-3\lambda)}$
Take $\frac{\lambda}{(1-2\lambda)(1-3\lambda)} = \frac{A}{(1-2\lambda)} + \frac{B}{(1-3\lambda)}$
$-3A - 2B = 1$
$A + B = 0 \to A = -1;\ B = 1$
$\frac{\lambda}{(1-2\lambda)\,(1-3\lambda)} = \frac{-1}{(1-2\lambda)} + \frac{1}{(1-3\lambda)}$
$\frac{\lambda^3}{(1-2\lambda)(1-3\lambda)} = -\frac{\lambda^2}{(1-2\lambda)} + \frac{\lambda^2}{(1-3\lambda)}$
$(BA)^{-1}\frac{\lambda^3}{(1-2\lambda)\,(1-3\lambda)} = -x + x^2$

Definition (5): convolution of Battor-AlTememe transform:
Let f and g be a function defined in [1, ∞] then the convolution of f and g is given by:
$(f * g)(x) = \lambda\int_1^x f(u)\,g\left(\frac{x}{u}\right)\frac{du}{u}$
Proof: similar to proof of AlTememe convolution [5], the difference is only we multiply the convolution of AlTememe transform by λ.
Note (1): If $BA(f(x)) = F(\lambda)$ and $BA(g(x)) = G(\lambda)$, then $BA(f * g) = F(\lambda)\,G(\lambda)$ [5].

Example (9): find $(BA)^{-1}\frac{\lambda^3}{(1-2\lambda)(1-3\lambda)}$
$(BA)^{-1}\frac{\lambda^3}{(1-2\lambda)(1-3\lambda)} = \frac{1}{\lambda}\cdot\frac{\lambda^4}{(1-2\lambda)(1-3\lambda)} = \frac{1}{\lambda}\cdot\frac{\lambda^2}{(1-2\lambda)}\cdot\frac{\lambda^2}{(1-3\lambda)}$
$BA(x) = \frac{\lambda^2}{1-2\lambda};\ \ BA(x^2) = \frac{\lambda^2}{1-3\lambda}$
$= \int_1^x u\left(\frac{x}{u}\right)^2\frac{du}{u} = x^2\int_1^x u^{-2}\,du = x^2(\frac{u^{-1}}{-1})]_1^x$
$= x^2(-x^{-1}+1) = -x + x^2$

Battor-AlTememe of derivatives

In this section we will introduce Battor-AlTememe of derivatives and this will simplify to solve the differential equations.

1) $BA(x\acute{y}) = -\lambda y(1) + \frac{1-\lambda}{\lambda}BA(y)$

Proof: $BA(x\acute{y}) = \lambda\int_1^\infty x^{-\frac{1}{\lambda}+1}\acute{y}\,dx = \lambda\left[x^{-\frac{1}{\lambda}+1}\right]_1^\infty - (-\frac{1}{\lambda}+1)\int_1^\infty x^{-\frac{1}{\lambda}}y\,dx\,]$

$$= -\lambda\,y(1) + \frac{1-\lambda}{\lambda}BA(y);\ \ \lambda \neq 0$$

2) $BA(x^2y'') = -\lambda y'(1) - (1-2\lambda)y(1) + \frac{(1-2\lambda)(1-\lambda)}{\lambda^2}BA(y)$

Proof: $BA(x^2y'') = \lambda\int_1^\infty x^{-\frac{1}{\lambda}+2}y''\,dx$
$= \lambda[x^{-\frac{1}{\lambda}+2}y']_1^\infty - (-\frac{1}{\lambda}+2)\int_1^\infty x^{-\frac{1}{\lambda}+1}y'\,dx\,] = -\lambda y'(1) + \frac{(1-2\lambda)}{\lambda}BA(xy') = -\lambda y'(1) - (1-2\lambda)y(1) + \frac{(1-2\lambda)(1-\lambda)}{\lambda^2}BA(y)$

3) $BA(x^3y''') = -\lambda y''(1) - (1-3\lambda)y'(1) - \frac{(1-3\lambda)(1-2\lambda)}{\lambda}y(1) + \frac{(1-3\lambda)(1-2\lambda)(1-\lambda)}{\lambda^3}BA(y)$

Proof: same as one and two.

4) $BA(x^4y^{IV}) = -\lambda y'''(1) - (1-4\lambda)y''(1) - \frac{(1-4\lambda)(1-3\lambda)}{\lambda}y'(1) - \frac{(1-4\lambda)(1-3\lambda)(1-2\lambda)}{\lambda^2}y(1) + \frac{(1-4\lambda)(1-3\lambda)(1-2\lambda)(1-\lambda)}{\lambda^4}BA(y)$

Proof: same as one and two.

So,
$BA\big(x^ny^{(n)}\big) = -\lambda y^{(n-1)}(1) - (1-n\lambda)\,y^{(n-2)}(1) - \frac{(1-n\lambda)(1-(n-1)\lambda)}{\lambda}\,y^{(n-3)}(1)\ldots + \frac{(1-n\lambda)(1-(n-1)\lambda)\ldots(1-\lambda)}{\lambda^n}\,BA(y)$

Proof: By Induction.

Example (10):

(1) To solve the ODE $xy' + y = 1\ ;\ y(1) = -2$
We take BA to both sides, we get
$$-\lambda\big(y(1)\big) + \frac{1-\lambda}{\lambda}\,BA(y) + BA(y) = \frac{\lambda^2}{1-\lambda}$$
BA(y) $= \frac{\lambda^3}{1-\lambda} - 2\lambda^2 = \frac{\lambda^2}{1-\lambda} - 3\lambda^2$ (By division)
$\Rightarrow y = 1 - 3x^{-1}\ (By\ inverse)$

(2) To solve the ODE
$x^2y'' + xy' + y = x\ ;\ y(1) = y'(1) = 1$
By taking BA to both sides, we get
$BA(y) = \frac{\lambda^4 - 2\lambda^3 + \lambda^2}{(2\lambda^2 - 2\lambda + 1)(1-2\lambda)}$
By partial fractions, we get
$BA(y) = \frac{\frac{1}{2}\lambda^2}{2\lambda^2 - 2\lambda + 1} + \frac{\frac{1}{2}\lambda^2}{1-2\lambda}$
By taking inverse to both sides, we get
$y = \frac{1}{2}\sin(\ln(x)) + \frac{1}{2}\cos(\ln(x)) + \frac{1}{2}x$
Since $(BA(\sin(\ln(x)) + \cos(\ln(x)))) = \frac{\lambda^2}{(1-\lambda)^2 + \lambda^2}$

(3) To solve the ODE $xy' - y = \ln(x)\ ;\ y(1) = 2$
After taking BA to both sides, we get
$BA(y) = \frac{\lambda^4}{(1-2\lambda)(1-\lambda)^2} + \frac{2\lambda^2}{1-2\lambda} = \frac{3\lambda^2}{1-2\lambda} - \frac{\lambda^2}{(1-\lambda)^2}$ (By taking partial fractions to first term)
By taking $(BA)^{-1}$ to both sides, we get
$y = 3x - \ln(x) - 1$ Since $BA(\ln(x) + 1) = \frac{\lambda^2}{(1-\lambda)^2}$

Rules: In the following we will give some useful rules

Rule (1): $BA(\ln(x)+1)=\frac{\lambda^2}{(1-\lambda)^2}$

Proof: L.S $BA(\ln(x)+1)=BA(\ln(x))+BA(1)=\frac{\lambda^3}{(1-\lambda)^2}+\frac{\lambda^2}{1-\lambda}=\frac{\lambda^3+\lambda^2(1-\lambda)}{(1-\lambda)^2}=\frac{\lambda^2}{(1-\lambda)^2}=R.S$

Rule (2): $BA(sin\,(\ln(x))+cos\,(\ln(x)))=\frac{\lambda^2}{(1-\lambda)^2+\lambda^2}$

Proof: L.S

$BA(sin\,(\ln(x))+cos\,(\ln(x)))=BA(sin\,(\ln(x)))+BA(cos\,(\ln(x)))=\frac{\lambda^3}{(1-\lambda)^2+\lambda^2}+=\frac{\lambda^2(1-\lambda)}{(1-\lambda)^2+\lambda^2}=\frac{\lambda^2}{(1-\lambda)^2+\lambda^2}=R.S$

Rule (3): $BA(sinh\,(ln(x))+cosh\,(ln(x)))=\frac{\lambda^2}{1-2\lambda}$

Proof: L.S

$BA(sinh\,(ln(x))+cosh\,(ln(x)))=BA(sinh\,(ln(x)))+BA(cosh\,(ln(x)))=\frac{\lambda^3}{(1-\lambda)^2-\lambda^2}+\frac{\lambda^2(1-\lambda)}{(1-\lambda)^2-\lambda^2}=\frac{\lambda^2}{1-2\lambda}=M.S=R.S$

Rule (4): $BA(nx^{n-1}-(n+1)x^n=\frac{-\lambda^2}{(1-n\lambda)(1-(n+1)\lambda)}\forall\quad n\,\epsilon\,R$

Proof: L.S $=n\frac{n\lambda^2}{1-n\lambda}-(n+1)\frac{\lambda^2}{1-(n+1)\lambda}=n\lambda^2[\frac{1}{1-n\lambda}-\frac{1}{1-(n+1)\lambda}]-\frac{\lambda^2}{1-(n+1)\lambda}=n\lambda^2[\frac{1-n\lambda-\lambda-1+n\lambda}{(1-n\lambda)(1-(n+1)\lambda)}]-\frac{\lambda^2}{1-(n+1)\lambda}=n\lambda^2[\frac{-\lambda}{(1-n\lambda)(1-(n+1)\lambda)}]-\frac{\lambda^2}{1-(n+1)\lambda}=\frac{\lambda^2}{1-(n+1)\lambda}[\frac{-n\lambda-1+n\lambda}{1-n\lambda}]=\frac{-\lambda^2}{(1-n\lambda)(1-(n+1)\lambda)}$

Example (11): To find $BA(1-2x)$

Here n = 1

$\Rightarrow=\frac{-\lambda^2}{(1-\lambda)(1-2\lambda)}$ from rule (4)

Now; $BA(1-2x)=BA(1)-2BA(x)$

$=\frac{\lambda^2}{1-\lambda}-2\frac{\lambda^2}{1-2\lambda}=\frac{\lambda^2-2\lambda^3-2\lambda^2+2\lambda^3}{(1-\lambda)(1-2\lambda)}=\frac{-\lambda^2}{(1-\lambda)(1-2\lambda)}$

Identities: In this section we will introduce number of identities which we are used to solve differential equations.

$$BA\big((x\ln(x))y'+y\big)=\frac{1-\lambda}{\lambda}BA(y\ln(x))$$

Proof: $BA\big((x\ln(x))y'+y\big)=\lambda\int_1^\infty x^{-\frac{1}{\lambda}+1}(\ln(x))y'dx$

$=\lambda\left(x^{-\frac{1}{\lambda}+1}(\ln(x))y]_1^\infty-\int_1^\infty x^{-\frac{1}{\lambda}}y\,dx-\left(-\frac{1}{\lambda}+1\right)\int_1^\infty x^{-\frac{1}{\lambda}}(\ln(x))y\,dx\right)$

$=-\lambda\int_1^\infty x^{-\frac{1}{\lambda}}y\,dx+\left(\frac{1-\lambda}{\lambda}\right).\lambda\int_1^\infty x^{-\frac{1}{\lambda}}(\ln(x))y\,dx$

$\Rightarrow\quad BA\big((x\ln(x))y'+y\big)=\frac{1-\lambda}{\lambda}BA(y\ln(x)$

Example (12): to solve the DE $\big((x\ln(x))y'+y=\ln(x)\big)$ by using the above identity

$\frac{1-\lambda}{\lambda}BA(y\ln(x))=\frac{\lambda^3}{(1-\lambda)^2}\quad\Rightarrow BA(y\ln(x))=\frac{\lambda^4}{(1-\lambda)^3}$

$\Rightarrow(y\ln(x))=\frac{1}{2}(\ln(x))^2$; By using $(BA)^{-1}$

$\Rightarrow y = \frac{1}{2}\ln(x)$

Example (13): to solve the DE $\big((x\ln(x))y' + y = 1\big)$ by using the above identity

$\frac{1-\lambda}{\lambda}BA(y\ln(x)) = \frac{\lambda^2}{1-\lambda} \quad \Rightarrow BA(y\ln(x)) = \frac{\lambda^3}{(1-\lambda)^2}$

$\Rightarrow (y\ln(x)) = \ln(x)$; (By using $(BA)^{-1}$)

$\Rightarrow y = 1$

1. $BA\big((x\sin(\ln(x)))y' + (\cos(\ln(x)))y\big) = \frac{1-\lambda}{\lambda}BA(\sin(\ln(x)))y)$

Proof: By finding $BA\big((x\sin(\ln(x)))y'\big)$ will get the result or proof.

Example (14): to solve the DE $(x\sin(\ln(x)))y' + (\cos(\ln(x)))y = 1$, by using the identity (2).

$\frac{1-\lambda}{\lambda}BA(\sin(\ln(x)))y) = \frac{\lambda^2}{1-\lambda}$

$\Rightarrow BA(\sin(\ln(x)))y) = \frac{\lambda^3}{(1-\lambda)^2}$

$\Rightarrow (\sin(\ln(x)))y = \ln(x)$ (By using $(BA)^{-1}$)

$\Rightarrow \ y = (\csc(\ln(x)))\ln(x)$

2. $BA\big((x\cos(\ln(x)))y' - (\sin(\ln(x)))y\big) = -\lambda y(1) + \frac{1-\lambda}{\lambda}BA((\cos(\ln(x)))y)$

To prove it get the first term $BA\big((x\cos(\ln(x)))y'\big)$ then will get the proof.

3. $BA\big((x(\sin(\ln(x)) + \cos(\ln(x)))y' + (\cos(\ln(x)) - \sin(\ln(x)))y\big) = -\lambda y(1) + \frac{1-\lambda}{\lambda}BA\big((\sin(\ln(x)) + \cos(\ln(x)))y\big)$

The idea of proof same as (2) and (3).

4. $BA\big((x(\sin(\ln(x)) - \cos(\ln(x)))y' + (\cos(\ln(x)) + \sin(\ln(x)))y\big) = -\lambda y(1) + \frac{1-\lambda}{\lambda}BA\big((\sin(\ln(x)) + \cos(\ln(x)))y\big)$

The idea of proof same as (2) and (3).

5. $BA\big((x\,sinh(ln(x))) + (cosh(ln(x)))y\big) = \frac{1-\lambda}{\lambda}BA\big((sinh(ln(x)))y\big)$
6. $BA\big((x\,cosh(ln(x)))y' + (sinh(ln(x)))y\big) = -\lambda y(1) + \frac{1-\lambda}{\lambda}BA\big((cosh(ln(x)))y\big)$
7. $BA\big((x(sinh(ln(x)) - cosh(ln(x)))y' + (cosh(ln(x)) - sinh(ln(x)))y\big) = -\lambda y(1) + \frac{1-\lambda}{\lambda}BA\big((sinh(ln(x)) - cosh(ln(x)))y\big)$
8. $BA\big((x(sinh(ln(x)) + cosh(ln(x)))y' + (cosh(ln(x)) + sinh(ln(x)))y\big) = -\lambda y(1) + \frac{1-\lambda}{\lambda}BA\big((sinh(ln(x)) + cosh(ln(x)))y\big)$
9. $BA(x(ln(x))^2y' + 2(ln(x))) = \frac{1-\lambda}{\lambda}BA((ln(x))^2y)$
10. $BA(x(ln(x))^3y' + 3(ln(x))^2y) = \frac{1-\lambda}{\lambda}BA((ln(x))^3y)$
11. $BA(x(ln(x))^ny' + n(ln(x))^{n-1}y) = \frac{1-\lambda}{\lambda}BA((ln(x))^ny)\ , \quad \forall\, n \in Z^+$

From above identities, we can get the following results:

i. From (6) $BA\big((x^2-1)y' + (x + x^{-1})y\big) = \frac{1-\lambda}{\lambda}BA\left(\frac{x^2-1}{x}y\right)$

ii. From (7) $BA\big((x^2+1)y' + (x - x^{-1})y\big) = -\lambda y(1) + \frac{1-\lambda}{\lambda}BA\left(\frac{x^2+1}{x}y\right)$

iii. From (8) $BA\left(-y' + \frac{1}{x}y\right) = -\lambda y(1) + \frac{1-\lambda}{\lambda}BA\left(-\frac{y}{x}\right)$

iv. From (9) $BA(x^2y' + xy) = -\lambda y(1) + \frac{1-\lambda}{\lambda} BA(xy)$

Examples (15):

1. To solve the DE $(x^2 - 1)y' + (x + x^{-1})y = 1$
   By taking BA to both sides of the DE and using first result (i) will get
   $\frac{1-\lambda}{\lambda} BA\left(\frac{x^2-1}{x} y\right) = \frac{\lambda^2}{1-\lambda}$
   $\Rightarrow BA\left(\frac{x^2-1}{x} y\right) = \frac{\lambda^3}{(1-\lambda)^2} \Rightarrow \frac{x^2-1}{x} y = \ln x$; by taking $(BA)^{-1}$
   $\Rightarrow y = \frac{x \ln x}{x^2-1}$; $x \neq \mp 1$

2. To solve $(x^2 + 1)y' + (x - x^{-1})y = \ln x; \ y(1) = 0$
   By taking BA to both sides of DE and by using result (ii) we get
   $y = \frac{\frac{1}{2} x (\ln x)^2}{x^2+1}$
3. To solve $x^2 y' + xy = 1; \ y(1) = 0$
   From result (iv) and by taking BA to both sides, we get
   $\frac{1-\lambda}{\lambda} BA(xy) = \frac{\lambda^2}{1-\lambda} \Rightarrow BA(xy) = \frac{\lambda^3}{(1-\lambda)^2}$
   By taking $(BA)^{-1} \Rightarrow xy = \ln x \Rightarrow y = x^{-1} \ln x$

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