Wiener index of Lehmer three mean graphs

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Abstract---The Wiener index W(G) of G is equal to the sum of distances between all pairs of vertices of G. The Wiener index W, denoted w (Wiener 1947) and also known as the path number or Wiener number, is a graph index defined by \( W(G) = \sum_{(x,y) \in V(G)} d_G(x,y) \). In this paper we investigate Wiener index for caterpillar, twig and arrow graph.

Keywords---Wiener index, Lehmer three mean graphs.

Introduction

The Wiener topological index \( (W) \), introduced around 1947 by Harry Wiener, is the representation of data through a network of vertices (nodes) and edges (connections) which construct shapes to interpret patterns and relationship properties." In graph theory, we define a simple connected graph, \( G \), with vertices, \( V \) and edges, \( E \) as \( G = (V, E) \). For \( x, y \in V \), the length of the shortest edge from \( u \) to \( v \) is represented as the distance, \( d(x,y) \). The Wiener Index, \( W(G) \), is the sum with respect to \( (x,y) \) of the subsets of \( G \).

Definition 1.1

Let \( G \) be a \((r,s)\) graph. A function \( h \) is called Lehmer three mean labeling of graph \( G \), if it is possible to label the vertices \( v \in V \) with distinct labels \( h(x) \) from \( 1,2,3,\ldots,s+1 \) in such a way that when each edge \( e = xy \) is labeled with \( h(e) = \)
\[ \frac{h(x)^3 + h(y)^3}{h(x)^2 + h(y)^2} \text{ (or) } \frac{h(x)^3 + h(y)^3}{h(x)^2 + h(y)^2} \] then the edge labels are distinct. In this case “\( h \)” is called Lehmer-3 mean labeling of \( G \).

**Definition 1.2**

Let \( G \) be a Lehmer three Mean graph. The Wiener Index \( W(G) \) of \( G \) is defined by

\[ W(G) = \sum_{(u,v) \in E(G)} d_G(u,v) \]

**Definition 1.3**

The average distance \( \mu(G) \) between the vertices of \( G \) by:

\[ \mu(G) = \frac{W(G)}{|V(G)|} \]

**Main Results**

**Theorem 2.1**

Twig \( Tw(m) \) be a Lehmer three mean graph, then Wiener index of \( Tw(m) \) is

\[ W(G) = (3m-5) \times 1 + 6 (m-2) \times 2 + \sum_{k=3}^{m-1} \sum_{j=3}^{m-1} 9(m-k)j \]

Average distance is

\[ \mu(G) = \frac{W(G)}{|V(G)|} \]

**Proof:**

Let Twig \( Tw(m) \) be a Lehmer three mean graph.

Then \( W(G) = \sum_{(u,v) \in E(G)} d_G(u,v) \)

\[ = \{ \{d(v_1, v_2) + d(v_1, u_1) + d(v_1, w_1) + d(v_1, v_3) + \ldots + d(v_1, v_m) + d(v_1, u_{m-2}) + d(v_1, w_{m-2}) \} + \{d(u_2, w_1) + d(u_1, v_3) + d(u_1, u_2) + \ldots + d(u_1, u_{m-2}) + d(u_1, w_{m-2}) + d(u_1, v_m) \} + \ldots + \{d(v_{m-1}, u_{m-2}) + d(v_{m-1}, w_{m-2}) + d(v_{m-1}, v_m) \} + d(u_{m-2}, v_m) \} \]

\[ = (3m-5) \times 1 + 6 (m-2) \times 2 + \sum_{k=3}^{m-1} \sum_{j=3}^{m-1} 9(m-k)j \]

Average distance is

\[ \mu(G) = \frac{W(G)}{|V(G)|} \]

\[ = \frac{(3m-5) \times 1 + 6 (m-2) \times 2 + \sum_{k=3}^{m-1} \sum_{j=3}^{m-1} 9(m-k)j}{2} \]

**Example 2.2**

Wiener Index of Twig graph \( Tw(4) \) is given below

\[ W(G) = \sum_{(u,v) \in E(G)} d_G(u,v) \]
\[ W(G) = (1 + 2 + 2 + 2 + 3 + 3 + 3 + 4 + 4 + 4) + (1 + 1 + 1 + 2 + 2 + 2 + 3 + 3 + 3) \\
+ (2 + 2 + 3 + 3 + 3 + 4 + 4 + 4) + (2 + 3 + 3 + 3 + 4 + 4 + 4) \\
+ (1 + 1 + 1 + 2 + 2 + 2) + (2 + 2 + 3 + 3 + 3) + (2 + 3 + 3 + 3) \\
+ (1 + 1 + 1) + (2 + 2) + 2 \\
= (10 \times 1) + (18 \times 2) + (18 \times 3) + (9 \times 4) \\
\mu(G) = \frac{136}{11C_2} = \frac{136}{55} \approx 2.5 \]

**Theorem 2.3**

Let Arrow graph \( A^2(m) \) be a Lehmer three mean graph, then Wiener index of \( A^2(m) \) is \( W(G) = 3m \times 1 + \sum_{k=1}^{m-1} \sum_{j=2}^{m} 4(m - k)j \). Average distance is \( \mu(G) = \frac{W(G)}{|V(G)|} \)

**Proof:**

Let Arrow graph \( A^2(m) \) be a Lehmer three mean graph. Then \( W(G) = \sum_{(u,v) \in E(G)} d_G(u,v) \)

\[
W(G) = 3m \times 1 + \sum_{k=1}^{m-1} \sum_{j=2}^{m} 4(m - k)j \\
= 3m \times 1 + \frac{m(m-1)}{2} \sum_{j=2}^{m} 4(m-j) \\
= 3m \times 1 + \frac{m(m-1)}{2} \sum_{j=2}^{m} 4m - 4j \\
= 3m \times 1 + \frac{m(m-1)}{2} \left[ (4m \times m) - (4 \times \sum_{j=2}^{m} j) \right] \\
= 3m \times 1 + \frac{m(m-1)}{2} \left[ 4m^2 - 4 \times \frac{m(m+1)}{2} \right] \\
= 3m \times 1 + \frac{m(m-1)}{2} \left[ 4m^2 - 2m(m+1) \right] \\
= 3m \times 1 + \frac{m(m-1)}{2} \left[ 2m(m-1) \right] \\
= 3m \times 1 + \frac{m(m-1)}{2} \frac{m(m-1)}{2} \\
= \frac{3m \times 1 + \sum_{k=1}^{m-1} \sum_{j=2}^{m} 4(m - k)j}{|V(G)|} \\
\]  

**Example 2.4**

Wiener Index of Arrow graph \( A^2(4) \) is shown below

\[ W(G) = \sum_{(u,v) \in E(G)} d_G(u,v) \]

\[
W(G) = (1 + 1 + 2 + 2 + 3 + 3 + 4 + 4) + (1 + 1 + 2 + 2 + 3 + 3 + 4) \\
+ (2 + 1 + 3 + 2 + 4 + 3) + (1 + 1 + 2 + 2 + 3) + (2 + 1 + 3 + 2) \\
+ (1 + 1 + 2) + (2 + 1) + 1 \\
= (12 \times 1) + (12 \times 2) + (8 \times 3) + (4 \times 4) \\
\mu(G) = \frac{76}{9C_2} = \frac{136}{36} \approx 3.7 \\
\]

**Theorem 2.3**

Let Caterpillar graph be a Lehmer three mean graph, then Wiener index of \( A^2(m) \) is \( W(G) = (3m - 1) \times 1 + 2(3m - 3) \times 2 + \sum_{k=2}^{m-1} \sum_{j=3}^{m} 3(3m - (3k - 1))j + 4(m+1) \)

**Proof:**

Let Caterpillar graph be a Lehmer three mean graph.
Then \( W(G) = \sum_{(u,v)\in V(G)} d_G(u,v) \)
\[
= \left[ (d(u_1,v_1) + d(u_1,w_1) + d(u_1,u_2) + \ldots + d(u_1,u_m) + d(u_1,v_m) + d(u_1,w_m)] + \\
(d(v_1,w_1) + d(v_1,u_2) + d(v_1,v_2) + \ldots + d(v_1,u_m) + d(v_1,v_m) + d(v_1,w_m) + \ldots + \\
+d(u_m,v_m) + d(u_m,v_m)) + d(v_m,w_m) \right] \\
= (3m - 1) \times 1 + 2(3m - 3) \times 2 + \sum_{k=2}^{m-1} \sum_{j=3}^{m} 3(3m - (3k - 1))j + 4(m + 1) \\
W(G) = (3m - 1) \times 1 + 2(3m - 3) \times 2 + \sum_{k=2}^{m-1} \sum_{j=3}^{m} 3(3m - (3k - 1))j + 4(m + 1)
\]

**Example 2.4**

Wiener Index of Caterpillar graph is shown below

\[
W(G) = \sum_{(u,v)\in V(G)} d_G(u,v) \\
W(G) = (1 + 1 + 1 + 2 + 2 + 2 + 3 + 3 + 3 + 4 + 4 + 4 + 5 + 5 + 5 + 6 + 6 + 6 + 7 + 7) \\
+ (2 + 2 + 3 + 3 + 3 + 4 + 4 + 4 + 5 + 5 + 5 + 5 + 6 + 6 + 6 + 7 + 7) \\
+ (1 + 1 + 1 + 2 + 2 + 3 + 3 + 4 + 4 + 5 + 5) \\
+ (2 + 2 + 3 + 3 + 4 + 4 + 5 + 5 + 5 + 5 + 6 + 6) \\
+ (2 + 3 + 3 + 3 + 4 + 4 + 5 + 5 + 6 + 5) \\
+ (1 + 1 + 1 + 2 + 2 + 2 + 3 + 3 + 4 + 4 + 5 + 5) \\
+ (2 + 2 + 3 + 3 + 3 + 4 + 4 + 5 + 5 + 5 + 5 + 6 + 5 + 6 + 6 + 7 + 7)
\]
\[
= (17 \times 1) + (30 \times 2) + (39 \times 3) + (30 \times 4) + (21 \times 5) + (12 \times 6) + (4 \times 7)
\]
\[
= 519
\]

Average distance is \( \mu(G) = \frac{W(G)}{|V(G)|} \)
\[
\mu(G) = \frac{519}{182} \approx 3.39
\]

**References**