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Degree coprime domination and degree non-coprime domination in graphs

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***Abstract*---**Let $G(V,E)$ be a finite, undirected, simple graph without isolated vertices. A dominating set D of $V(G)$ is called a *degree coprime dominating set* of G if for every $v \in V - D$, there exist a vertex $u \in D$ such that $uv \in E(G)$ and $(deg\ u, deg\ v) = 1$. The minimum cardinality of a degree coprime dominating set is called the *degree coprime domination number* of G and is denoted by $\gamma_{cp}(G)$. A dominating set D of $V(G)$ is called a *degree non-coprime dominating set* of G if for every $v \in V - D$, there exist a vertex $u \in D$ such that $uv \in E(G)$ and $(deg\ u, deg\ v) \neq 1$. The minimum cardinality of a degree non-coprime dominating set is called the *degree non-coprime domination number* and is denoted by $\gamma_{ncp}(G)$. We obtain the degree coprime domination number for some graphs.

***Keywords*---**degree coprime dominating set, degree non-coprime dominating set, degree coprime domination number, degree non-coprime domination number.

Introduction

Let $G(V,E)$ be a finite, undirected, simple graph without isolated vertices. The domination is the most interesting concept in graph theory. Berge introduce the concept domination number. In 1962, O Ore used the same concept, dominating set and domination number in graph theory. In 1970's more researchers worked on that concept besides to improve. As their result, there are more types of dominating sets [2] till now. There are so many applications using the concept

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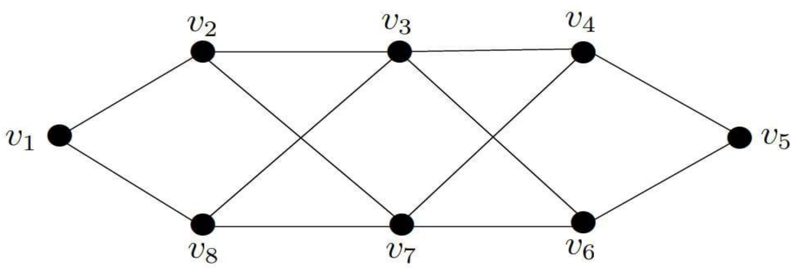
dominating set and domination number. Let $G = (V, E)$ be an undirected simple graph. A subset D of vertices of G is a dominating set if every vertex in $V - D$ is adjacent to atleast one vertex in D. The degree of a vertex v of a graph G is number of edges incident with v and is denoted by $deg\ v$. If all vertices of a graph G have the same degree then G is called regular graph. In this paper we introduce the concept, degree coprime dominating set and degree coprime domination number. Due to the comparison of degree D and $V - D$ the dominating set shrinked to the coprime dominating set. Further we refer [1, 3, 4]

Degree coprime dominating set

Definition 2.1 A dominating set D of a graph G is called a degree coprime dominating set of G if for every $v \in V - D$, there exist a vertex $u \in D$ such that $uv \in E(G)$ and $(\deg u, \deg v) = 1$. The minimum cardinality of a degree coprime dominating set is called the degree coprime domination number of G and is denoted by $\gamma_{cp}(G)$.

Definition 2.2 A degree coprime dominating set D is said to be minimal if no proper subset of D is a degree coprime dominating set of G.
Example: Consider a graph G as follows,



The minimal degree coprime dominating sets are $\{v_1, v_3, v_5, v_7\}$ and $\{v_2, v_4, v_6, v_8\}$, $\gamma_{cp}(G) = 3$.

Observation: We observe that the degree coprime domination number of any graph G is lies between 1 and $|V(G)|$. That is, $1 \leq \gamma_{cp}(G) \leq n$.

Theorem 2.3 *Let P_n be a path of order n. Then $\gamma_{cp}(P_n) = n - 2$, if $n \geq 3$.*
Proof: Let P_n be a path of order n and $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ such that $v_i v_{i+1} \in E(P_n)$ for all $i = 1,2,3, \ldots, n-1$. Then $deg\ v_1 = deg\ v_n = 1$ and $deg\ v_i = 2$ for $i = 2,3, \ldots, n-1$. If $n = 3$ then $\gamma_{cp}(P_3) = 1 = n - 2$. Let $D = \{v_2, v_3, \ldots, v_{n-1}\}$ be a dominating set. Since $(deg\ v_i, deg\ v_1) = (deg\ v_i, deg\ v_n) = (2,1) = 1$ for all $v_i \in D$, D is coprime dominating set of P_n. $\gamma_{cp}(P_n) \leq |D| = n - 2$. Suppose if $\gamma_{cp}(P_n) < n - 2$. Let D be a $\gamma_{cp} - set$ of P_n. Then there exist $u \in V - D$ such that $deg\ u = 2$, since $n \geq 4$. Clearly $(deg\ u, deg\ v) \neq 1$ for any $v \in D$, which is contradiction. Hence $\gamma_{cp}(P_n) = n - 2$.

Remark 2.4 *If $n = 2$ then $\gamma_{cp}(P_n) = \gamma_{cp}(K_2) = 1$.*

Theorem 2.5 *Let W_n be a wheel of order n.*

Then $\gamma_{cp}(W_n) = \begin{cases} 1 & \text{if}(n-1,3)=1 \\ n & \text{if}(n-1,3)\neq 1. \end{cases}$

Proof: Let W_n be a wheel of order n. Let $V(W_n)=\{v_1,v_2,\dots,v_n\}$ such that $deg\ v_1 = n-1$. Then $deg\ v_i = 3$ for all $i=1,2,\dots,n$.

Case 1: If $(n-1,3)=1$

Let $D=\{v_1\}$ be a dominating set. Then $(deg\ v_1, deg\ v_i)=(n-1,3)=1$ for all $v_i \in V-D$. So that D is degree coprime dominating set of W_n. Thus $\gamma_{cp}(W_n) \leq |D| = 1$. Hence $\gamma_{cp}(W_n)=1$.

Case 2: If $(n-1,3)\neq 1$

Clearly $\gamma_{cp}(W_n)\leq n$. Suppose $\gamma_{cp}(W_n)<n$. Let D be $\gamma_{cp}-set$ of W_n. Then $V-D$ has atleast one vertex u such that $deg\ u = n-1$ or $deg\ u=3$.

Sub case i: If $deg\ u=n-1$ then there exist $v\in D$ such that $deg\ v=3$. This implies that $(deg\ u, deg\ v)\neq 1$, which is contradiction.

Sub case ii: If $deg\ u=3$ then there exist $v \in D$ such that $deg\ v = 3$ or $n-1$. This implies that $(deg\ u, deg\ v)\neq 1$, which is contradiction. Hence $\gamma_{cp}(W_n)=n$.

Theorem 2.6 *Let F_n be a fan of order $n(n\geq 4)$.*

Then $\gamma_{cp}(F_n) = \begin{cases} n, & \text{if}(n-1,3)=3\ and\ (n-1,2)=2 \\ 2, & \text{if}(n-1,3)\neq 3; (n-1,2)=2\ and\ n=5 \\ 3, & \text{if}(n-1,3)\neq 3; (n-1,2)=2\ and\ n>5 \\ n-2, & \text{if}(n-1,3)=3\ and\ (n-1,2)\neq 2 \\ 1, & \text{otherwise.} \end{cases}$

Proof: Let F_n be a fan of order $n(n\geq 4)$. Let $V(F_n)=\{v_1,v_2,\dots,v_n\}$ such that $deg\ v_1 = n-1$ and $v_iv_{i+1}\in E(G)$ for all $i=2,3,\dots,n-1$. Then $deg\ v_i=2$ for all $i=1,n$ and $deg\ v_i = 3$ for all $i=3,4,\dots,n-1$.

Case 1: If $(n-1,3)=3$ and $(n-1,2)=2$

Clearly $\gamma_{cp}(F_n)\leq n$. Suppose $\gamma_{cp}(F_n)<n$. Let D be a $\gamma_{cp}-set$ of F_n. Then there exist atleast one vertex $u\in V-D$ such that $deg\ u=n-1$ or $deg\ u=3$ or $deg\ u=2$. Then there exist $v\in D$ such that $(deg\ u, deg\ v)\neq 1$, which is contradiction to D is $\gamma_{cp}-set$ of F_n. Hence $\gamma_{cp}(F_n)=n$.

Case 2: If $(n-1,3)\neq 3$; $(n-1,2)=2$ and $n=5$

Let $D=\{v_3,v_4\}$ be a dominating set. Since $(deg\ v_i, deg\ v_j)=1$ for all $v_i\in D$ and $v_j\in V-D$, D is degree coprime dominating set of F_5.

Case 3: If $(n-1,3)\neq 3$; $(n-1,2)=2$ and $n>5$

Let $D=\{v_1,v_2,v_n\}$ be a dominating set. Since $deg\ v=3$ for all $v\in V-D$, $(deg\ u, deg\ v)=1$, for all $u\in D$. Therefore D is degree coprime dominating set. Thus $\gamma_{cp}(F_n)\leq |D|=3$. Suppose $\gamma_{cp}(F_n)<3$. let D be a $\gamma_{cp}-set$ of F_n.

Sub case i: If $v_1\in D$ then there exist another vertex $u\in D$ such that either $deg\ u = 2$ or $deg\ u=3$. Hence there exist a vertex $v\in V-D$ such that either $deg\ v=2$ or $deg\ v=3$ respectively. This implies that $(deg\ u, deg\ v)\neq 1$, which is a contradiction to D is $\gamma_{cp}-set$ of F_n.

Sub case ii: If $v_1 \notin D$ then there exist a vertex $v\in V-D$ such that no vertex in D can dominates v, since $n>5$. Therefore D is not dominating set, which is contradiction. Hence $\gamma_{cp}(F_n)=3$.

Case 4: If $(n-1,3)=3$ and $(n-1,2)\neq 2$
If $n=4$ then the $\gamma_{cp}-sets$ are $\{v_1,v_3\}$ and $\{v_2,v_4\}$. Hence $\gamma_{cp}(F_4)=2=n-2$. If $n>4$, let $D=\{v_1,v_3,v_4,\ldots,v_{n-1}\}$ be a dominating set. Since $(deg\ v_i, deg\ v_j)=1$ for all $v_i\in D$ and $v_j\in V-D$, D is degree coprime dominating set of F_n. Therefore $\gamma_{cp}(F_n)\leq n-2$. Suppose $\gamma_{cp}(F_n)<n-2$. Let D be a $\gamma_{cp}-set$ of F_n. If $v_1\in D$ then there exist a vertex $v\in V-D$ such that $deg\ v=3$. If $v_1\notin D$ then there exist a vertex $v\in D$ such that $deg\ v=3$. Both cases are leads to the contradiction that $(deg\ v_1, deg\ v)=3\neq 1$. Therefore $\gamma_{cp}(F_n)=n-2$.

Case 5: If $(n-1,3)\neq 3$ and $(n-1,2)\neq 2$
Let $D=\{v_1\}$ be a dominating set such that $deg\ v_1=n-1$. Since $(deg\ v_1, deg\ v_i)=(n-1,2)=1$ for all $i=2,n$ and $(deg\ v_1, deg\ v_i)=(n-1,3)=1$ for all $i=3,4,\ldots,n-1$, D is a degree coprime dominating set. Hence $\gamma_{cp}(F_n)=1$.

Theorem 2.7 *Let* $K_{m,n}(2\leq m\leq n)$ *be a complete bipartite graph of order* $m+n$. Then $\gamma_{cp}(K_{m,n})=\begin{cases}2, & \text{if}(m,n)=1\\ m+n, & \text{if}(m,n)\neq 1\end{cases}$.
Proof: Let $K_{m,n}$ be a complete bipartite graph with bipartition (U,V) where $U=\{u_1,u_2,\ldots,u_n\}, V=\{v_1,v_2,\ldots,v_n\}$. Then $deg\ u_i=n, deg\ v_j=m$ for all $u_i\in U$ and $v_j\in V$.

Case 1: If $(m,n)=1$
Let $D=\{v_k,v_l\}$, for some $1\leq k\leq m, 1\leq l\leq n$. Clearly, D is a dominating set, then $(deg\ u_k, deg\ v_j)=(n,m)=1$ for all $v_j\in V-D$ and $(deg\ v_l, deg\ u_i)=1$ for all $u_i\in V-D$. So that D is degree a coprime dominating set of $K_{m,n}$. Thus $\gamma_{cp}(K_{m,n})\leq|D|=2$. Suppose $\gamma_{cp}(K_{m,n})<2$. Let D be a $\gamma_{cp}-set$ of $K_{m,n}$. Then either $D=\{u\}$ or $D=\{v\}$ for $u\in U$ and $v\in V$. Then there exist a vertex v either in U or in V such that no vertex of D can be dominate the vertex v. Clearly, D is not a dominating set. Thus $\gamma_{cp}(K_{m,n})\geq 2$. Hence $\gamma_{cp}(K_{m,n})=2$.

Case 2: If $(m,n)\neq 1$
Clearly, $\gamma_{cp}(K_{m,n})\leq m+n$. Suppose if $\gamma_{cp}(K_{m,n})<m+n$, let D be a $\gamma_{cp}-set$ of $K_{m,n}$. Then $V-D$ has atleast one vertex u such that $deg\ u=m$ or $deg\ u=n$. Then there exist $v\in D$ such that $(deg\ u, deg\ v)\neq 1$, which is contradiction. Hence $\gamma_{cp}(K_{m,n})=m+n$.

Theorem 2.8 *Let* $K_{1,n}$ *be a star of order* $n+1$*. Then* $\gamma_{cp}(K_{1,n})=1$.
Proof: Let $K_{1,n}$ be a star of order $n+1$ and $V(K_{1,n})=\{v_1,v_2,\ldots,v_{n+1}\}$ such that $deg\ v_1=n$. Let $D=\{v_1\}$ be a dominating set. Since $(deg\ v_1, deg\ v_i)=1$ for all $v_i\in V-D$, D is a degree coprime dominating set of $K_{1,n}$. Hence $\gamma_{cp}(K_{1,n})=1$.

Theorem 2.9 *Let* G *be any graph order* n*. Then* $\gamma_{cp}(G\circ K_1)=n$.
Proof: Let G be any graph of order n. $G\circ K_1$ be product of a graph G and K_1. Let $V(G\circ K_1)=\{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n\}$ such that $deg\ v_i=1$ for all $i=1,2,\ldots,n$. Let $D=\{v_1,v_2,\ldots,v_n\}$ be a dominating set. Since $(deg\ v_i, deg\ u_j)=1$ for all $v_i\in D$ and $u_j\in V-D$, D is a degree coprime dominating set of $G\circ K_1$. Therefore $\gamma_{cp}(G\circ K_1)\leq n$. Suppose $\gamma_{cp}(G\circ K_1)<n$. Let D be a $\gamma_{cp}-set$ of $(G\circ K_1)$. Then there exist a vertex $v\in V-D$ ($v\in V(G)$ or $v\in V(G\circ K_1)-V(G)$) such that no vertex of D can be dominate

the vertex v, which is a contradiction to D is a $\gamma_{cp} - set$ of $(G \circ K_1)$. Hence $\gamma_{cp}(G \circ K_1) = n$.

Theorem 2.10 *If G is regular graph of order $n(n > 2)$ then $\gamma_{cp}(G) = n$.*
Proof: Let G be a regular graph of order n. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ such that $deg\ v_i = k$, for all $i = 1,2, \dots, n$. Clearly, $\gamma_{cp}(G) \leq n$. Suppose $\gamma_{cp}(G) < n$, let D be a $\gamma_{cp} - set$ of G then there exist $u \in V - D$ such that $uv \in E(G)$ for $v \in D$. Therefore $(deg\ v, deg\ u) = k$, which is contradiction. Hence $\gamma_{cp}(G) = n$.

Corollary 2.11 *Let C_n be a cycle of order n. Then $\gamma_{cp}(C_n) = n$.*
Corollary 2.12 *Let K_n be a complete graph of order n. Then $\gamma_{cp}(K_n) = n$.*

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