How to Cite:
Abbas, N. S., & Bhaya, E. S. (2022). Saturation problems for positive linear approximation of function in quasi normed spaces. *International Journal of Health Sciences*, *6*(S3), 12070–12077. https://doi.org/10.53730/ijhs.v6nS3.9083

Saturation problems for positive linear approximation of function in quasi normed spaces

Nada Sadiq Abbas
Ministry of Education, General Directorate of Education in Babylon, Iraq
Email: nadasadiq41@gmail.com

Eman Samir Bhaya
University of Babylon, College of Education for Pure Sciences, Babilon, Iraq
Email: emanbhaya@itnet.uobabylon.edu.iq

***Abstract*---**Many authors work on constrained approximation such as monotonicity, convexity and k-monotoni city, but little works introduced in positive collinear approximation. The aim of our paper is to investigate linear and positive approximation for real functions in L_(P[0,1]), saturation problem between degree of best positive and linear approximation, and find collinear positive best approximation for measurable function in L_(P[0,1]) Lebesgue quasi normed spaces

***Keywords*---**saturation, positive, approximation, quasi normed spaces.

Introduction

In [6] the authors proved the following direct estimate for Bernstein operator:

$$|f(x)-(B_nf)(x)| \le C\omega^2_{\varphi^\lambda}\left(f, n^{-\frac{1}{2}}, \varphi(x)^{1-\lambda}\right) \qquad x \in I=[0,1] \quad (1)$$

in which $\varphi:[0,1] \to R$ is an admissible step weight function for details about φ see [9].

If $\lambda = 0$ in (3.1) we get classical local estimate while when λ=1 we get global norm estimate developed by Ditzian and Totik. So (3.1) fill the gap between the local and global approximation theorems for the Bernstein operator. Such result for polynomial approximation for details see [7,8 and 14]

Inequality (3.1) show that the error $f(x)-(B_nf)(x)$ is bounded pointwise by

Manuscript submitted: 18 Feb 2022, Manuscript revised: 27 April 2022, Accepted for publication: 9 June 2022

$C\left(n^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\right)^{\alpha}$ if $\omega^2_{\varphi^\lambda}(f,\delta)=O(\delta^\alpha)$ and $\alpha\in[0,2]$.

In [5,16] the authors proved the converse result also true.
$\omega^2_{\varphi^\lambda}(f,\delta)=0(\delta^\alpha)$ can be estimated in terms of the Bernstein operator that is the equivalence

$$|f(x)-(B_nf)(x)|=O\left(\left(f,n^{-\frac{1}{2}},\varphi(x)^{1-\lambda}\right)^{\alpha}\right)\Leftrightarrow\ \omega^2_{\varphi^\lambda}(f,\delta)=O(\delta^\alpha)$$

Holds whenever $\alpha\epsilon(0,2)$ and $\lambda\in[0,1]$.
We mean by $g=O(f)$ that $g\le c(f)$ for some constant C.
in [10] (1) can be considered as a further estimate for the general estimate

$$|f(x)-(B_nf)(x)|\le C\omega^2_{\varphi^\lambda}\left(f,n^{-\frac{1}{2}}\,\frac{\varphi(x)}{\emptyset(x)}\right)\quad(2)$$

$x\in[0,1]$ where $\varphi:[0,1]\to R$ is an admissible step_weight function of the Ditzian _Totik modulus [9] and $\emptyset^2$ is a concave function.
(2) improve meet of (1) if $\emptyset$ is replaced by $\varphi^\lambda,\lambda\in[0,1]$.

In [10] the authors proved inverse result to (2) for $x\in(0,2)$ $i.e.$
$|f(x)-(B_nf)(x)|\le C_1\left(n^{-\frac{1}{2}}\ \frac{\varphi(x)}{\emptyset(x)}\right)^{\alpha}$, $x\in[0,1]$ $n=1,2,3,\dots..$
Implies $\omega^2_{\varphi^\lambda}(f,\delta)\le C_2(\delta^\alpha)$ if, in addition $\varphi^2/\emptyset^2$ is concave which is satisfied for $\emptyset=\varphi^\lambda$, $\lambda\in[0,1]$ in particular.

Auxiliary Results

Here we introduce our auxiliary results that we need in our proof. We can modify a result in [15] to obtain the following Lemma.

In [15], the author used g_n as a positive function with integral equal to 1 and $\emptyset$ is any weight function.

Lemma 2.1 [15]

$$\int_0^1\frac{d^2}{dx^2}\,g_n(x,u)\le c(p)\left(4+\frac{(\emptyset^2(x))''}{n}+2\left(\frac{n^2}{\emptyset^4(x)}\int_0^1\left[\frac{(\emptyset^2(x))'}{n}-(u-x)\right](u-x)^3g_n(x,u)du\right)\right)\le M$$

and M must be an absolute constant independent of n and x.

Lemma 2.2 [15]

$$\left|\int_x^t g''(s)(t-s)ds\right| \leq \left\|\emptyset^2 g''\right\|_p \frac{(t-x)^2}{\emptyset^2(x)}$$

Lemma 2.3 [9]

For $f \in L_p(J)$ $,0 < p \leq \infty$ we have

$$\omega_\varphi^2(f,\delta)_p \approx K_\varphi^2(f,\delta^2)_p$$

Lemma 2.4[6]

$$\Delta_h^2(f)(x) = \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-h}{2}}^{\frac{h}{2}} f''\ (x+s+t)ds\, dt$$

The Main Results

In this paper we prove direct estimates for the approximation of function in L_P $,0 < p < 1$ space using positive linear operator. and for pointwise linear operator. Also, we estimate inverse result for positive linear approximation.
Define

$$w_\varphi(f,\delta)_p = \sup_{|h| \leq \delta} \left\| f(x+\varphi(x)\tfrac{h}{2} - f(x-\varphi(x)\tfrac{h}{2}\right\|_{L_{P(I)}}$$

$$I = [0,1]$$

Where $x+\varphi(x)\frac{h}{2}$ $, x+\varphi(x)\frac{h}{2} \in I$

$$\widehat{\omega}_\varphi(f,\delta)_p = \sup_{|h| \leq \delta} \|f(x+\varphi(x)h - f(x)\|_{L_{P(I)}}$$

$$x\ ,\ x+\varphi(x)h \in I$$

For the equivalence of $w_\varphi(f,\delta)_p$ and $\widehat{\omega}_\varphi(f,\delta)_p$ see [D.T] [9]
Now let us prove our first main result:

Theorem 3.1

Let $f \in L_{P[0,1]}$ and $\emptyset: [0,1] \rightarrow R$ be step function let
$A_n: L_{P[0,1]} \rightarrow L_{P[0,1]}$ $,n \in N$ be bounded positive linear operators so that

$$\left\|(A_n f)''(x)\right\|_p \leq c(p)\frac{n}{\varphi^2(x)}\|f\|_p \quad , \qquad (3)$$

for $x \in [0,1]$ $,f \in L_{P[0,1]}$, and

$$\left\|\emptyset^2(A_n g)''\right\|_p \leq c(p)\left\|\emptyset^2 g''\right\|_p \quad , \qquad (4)$$

where $\emptyset^2, \varphi^2$ and $\varphi^2/\emptyset^2$ are concave function on $[0,1]$ and $\alpha \in (0,2)$. Then for $f \in L_{P[0,1]}$,the pointwise approximation

$$\|f-(A_n f)(x)\|_p \leq c(p)\left(n^{-\frac{1}{2}}\frac{\varphi(x)}{\emptyset(x)}\right)^\alpha \qquad n = 1,2,.. \quad . \quad (5)$$

implies

$$\omega_\varphi^2(f,\delta)_p \leq c(p)\delta^\alpha \qquad \delta > 0.$$

Proof
Let $x, h \in [0,1]$ so that $x + h \in [0,1]$ and let

$$(\Delta_h^2 f)(x) = f(x+h) - 2f(x) + f(x-h)\,.$$

and
$\|(\Delta_h^2 f)(x)\|_p = \|\Delta_h^2 (f - A_n + A_n)(x)\|_p$

$$=\left\|\left((\Delta_h^2 (f - A_n f)\right)(x) + (\Delta_h^2 A_n f)(x)\,\right\|_p$$

$$\leq 2^{\frac{1}{p}} \left(\int_0^1 |(\Delta_h^2 (f - A_n f)(x)|^p\right)^{\frac{1}{p}} + 2^{\frac{1}{p}} \left(\int_0^1 |(\Delta_h^2 A_n f)(x)|^p\right)^{\frac{1}{p}}. \qquad (6)$$

$$\left\|\left((\Delta_h^2 (f - A_n f)\right)(x)\right\|_p = \|(f - A_n)(x+h) - 2(f - A_n)(x) + (f - A_n)(x-h)\|_p$$

$$\leq\ 2^{\frac{1}{p}-1} \left(\|(f - A_n)(x+h)\|_p + 2\|(f - A_n)(x)\|_p + \|(f - A_n)(x-h)\|_p\right)$$

Using (5) we obtain

$$\left\|\left((\Delta_h^2 (f - A_n f)\right)(x)\right\|_p\ \leq 2^{\frac{1}{p}-1} \left(\left(\frac{\varphi(x+h)}{\emptyset(x+h)}\right)^\alpha + 2\left(\frac{\varphi(x)}{\emptyset(x)}\right)^\alpha + \left(\frac{\varphi(x-h)}{\emptyset(x-h)}\right)^\alpha\right) n^{-\frac{\alpha}{2}}$$

$$\leq\ c(p) n^{-\frac{\alpha}{2}} \left(\frac{\varphi(x)}{\emptyset(x)}\right)^\alpha\ . \qquad (7)$$

Using Lemma 2.3 $\omega_\emptyset^2(f,\delta)_p$ and $K_\varphi^2(f,\delta^2)_p$ are equivalent a

$$c_2\ K_\varphi^2(f,\delta^2)_p \leq \omega_\emptyset^2(f,\delta)_p \leq c_1\ K_\varphi^2(f,\delta^2)_p$$

where c_1 and c_2 are absolute constants.

$$K_\varphi^2(f,\delta^2)_p = \|f - g\|_p + \delta^2 \left\|\emptyset^2 g''\right\|_p \leq c\omega_\emptyset^2(f,\delta)_p$$

$\|f - g\|_p\ \leq A\omega_\emptyset^2(f,\delta)_p\,,\qquad \left\|\emptyset^2 g''\right\|_p \leq B\,\delta^{-2}\omega_\emptyset^2(f,\delta)_p\,.$

From (3) and (4) we obtain

$$\left\|(A_n f)''(y)\right\|_p\ \leq \left\|A_n''(f - g)(y) + (A_n'' g)(y)\right\|_p$$

$$\leq 2^{\frac{1}{p}}\ \left(\int_0^1 \left|A_n''(f - g)(y)\right|^p\right)^{\frac{1}{p}} +\ 2^{\frac{1}{p}}\ \left(\int_0^1 \left|(A_n'' g)(y)\right|^p\right)^{\frac{1}{p}}$$

$$\leq c(p)\ \frac{n}{\varphi^2(y)} \|f - g\|_p + c(p)\ \frac{1}{\emptyset^2(y)}\quad \left\|\emptyset^2 g''\right\|_p$$

$$= c(p) \frac{n}{\varphi^2(y)} \|f - g\|_p + c(p)\ \frac{\delta^{-2}\,\delta^2}{\emptyset^2(y)}\quad \left\|\emptyset^2 g''\right\|_p$$

$$\leq c(p)\ \left(\frac{n}{\varphi^2(y)} + \frac{1}{\emptyset^2(y)\delta^2}\right) K_\varphi^2(f,\delta^2)_p$$

$$= c(p)\ \left(\frac{n}{\varphi^2(y)} + \frac{1}{\emptyset^2(y)\delta^2}\right) \omega_\emptyset^2\,(f,\delta)_p \qquad (\ 8)$$

Using Lemma 2.4, we obtain

$$\|(\Delta_h^2\, A_n f)(x)\|_p = \left\| \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} (A_n^{"} f)(x+s+t) ds dt \right\|_p .$$

Since A_n is bounded operators

$$= \left(\int_0^1 \left| \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} (A_n^{"} f)(x+s+t) ds dt \right|^p dx \right)^{\frac{1}{p}}$$

$$\|(\Delta_h^2\, A_n f)(x)\|_p \le c \left(\int_0^1 \left| h^2\, A_n^{"}(f)(x) \right|^p \right)^{\frac{1}{p}},$$

Then using (8) we obtain

$$\|(\Delta_h^2\, A_n f)(x)\|_p \le c(p) h^2 \left(\frac{n}{\varphi^2(x)} + \frac{1}{\delta^2 \emptyset^2(x)} \right) \;\; \omega_\emptyset^2\,(f,\delta)_p .$$

Using (6) we get

$$\|(\Delta_h^2\, A_n f)(x)\|_p \le \;\; c(p) \left(\frac{\varphi(x)}{\emptyset(x)} \right)^{\alpha} n^{-\frac{\alpha}{2}} + \left(\frac{nh^2}{\varphi^2(x)} + \frac{h^2}{\delta^2 \emptyset^2(x)} \right) \;\; \omega_\emptyset^2\,(f,\delta)_p$$

This implies

$$\omega_\emptyset^2\,(f,\delta)_p \le c(p) \left(\delta^{\frac{\alpha}{2}} + (\delta + \frac{h^2}{\delta^2}) \right)$$
$$\le c(p)\; \delta^{\alpha} .$$

Notations and Defines

Define the sequence of operators:

$$Q_n\,(f(x)) = \int_a^b g_n(u,x) f(u) du \qquad , a,b \in R,\;\; n \in N \qquad (9)$$

Such that

$g_n(u,x)$ is positive for any $x \in [a,b]$
and
$\int_a^b g_n(u,x)\, du = 1$ for any $n \in N$

Also, we need to define

$$\frac{d}{dx}\, g_n(u,x) = \frac{n}{\varphi^2}\, g_n(u,x)\,(u-x) \qquad , n \in N$$

where $\emptyset$ is a step weight function.

we use $L^2_{P[0,1]} = \{f: [a,b] \to R : f, f'' \in L_{p[0,1]}\}$ which is called $2-$fold L_p space

Proposition 3.2

Lf $g \in L^2_{P[0,1]}$, then for every function $\emptyset : [0,1] \to R$ there exists $\mathrm{M} > 0$

$$\left\|\emptyset^2 (\mathrm{G_n g})''\right\|_p \le \mathrm{M}\left\|\emptyset^2\, \mathrm{g}''\right\|_p \quad, \mathrm{n} \in \mathrm{N}$$

Where M is an absolute constant.

Proof
By Taylor's formula we have
$(\mathrm{G_n g})''$

$$= \int_0^1 \left[\frac{d^2}{dx^2}\ g_n(u,x)\right] \left(g(x) + g'(x)(u-x) \quad + \int_x^u (u-s) g''(x) ds\right) du$$

$$= \int_0^1 \left[\frac{d^2}{dx^2}\ g_n(u,x)\right] \left(g(x) + g'(x)(u-x)\right) du + \int_0^1 \left[\frac{d^2}{dx^2}\ g_n(u,x)\right] \left(\int_x^u (u-s) g''(x) ds\right) du$$

Because

$$\int_0^1 \left[\frac{d^2}{dx^2}\ g_n(u,x)\right] (u-x)^i du = 0 \qquad for\ \ i = 0,1$$

we obtain

$$(\mathrm{G_n g})'' = \int_0^1 \left[\frac{d^2}{dx^2}\ g_n(u,x)\right] \int_x^u (u-s) g''(x) ds\ \ du.$$

$$\left\|\emptyset^2 (\mathrm{G_n g})''\right\|_p = \left\|\emptyset^2(x) \int_0^1 \frac{d^2}{dx^2} g_n(u,x) \int_x^u (u-s) g''(x) ds du\right\|_P$$

$$= \left\|\emptyset^2(x) \int_0^1 \frac{d^2}{dx^2} g_n(u,x) \int_x^u (u-s) g''(x) ds du\right\|_p$$

$$\le \left(\int_0^1 \left(|\emptyset^2(x)| \int_0^1 \left|\frac{d^2}{dx^2} g_n(u,x)\right| \left|\int_x^u (u-s) g''(x) ds\right| du\right)^p dx\right)^{1/p}$$

By using Lemma2.2 we get

$$\left\|\emptyset^2 (\mathrm{G_n g})''\right\|_p \le \left(\int_0^1 \left(|\emptyset^2(x)| \int_0^1 \left|\frac{d^2}{dx^2} g_n(u,x)\right| \left| \emptyset^2(x) g''(x) \frac{(u-s)^2}{\emptyset^2(x)}\right| du\right)^p dx\right)^{1/p}$$

$$\le \left(\int_0^1 \left(\left|\emptyset^2(x)\, g''(x)\right| \int_0^1 \left|\frac{d^2}{dx^2} g_n(u,x)\right| \ |(u-s)^2\ | du\right)^p dx\right)^{1/p}$$

Using Lemma 2.1 and bounded ness of g'',we
Obtain

$$\left\|\emptyset^2(\mathrm{G_n g})^{"}\right\|_{\mathrm{p}} \leq M\left(\int_0^1 \left(\left|\emptyset^2(x)g^{"}(x)\right|\right)^p dx\right)^{1/p}$$
$$= M\left\|\emptyset^2 g^{"}\right\|_p$$

Theorem 3.3

Let G_n , $n \in N$,be exponential type operator

$$(G_n f) = \int_0^1 g_n(u,x)f(u)du \qquad , n \in N$$

Lf $\emptyset^2, \varphi^2$ and $\frac{\varphi^2}{\emptyset^2}$ are concave function on $[0,1]$ then for $f \in L_{p[0,1]}$ and $0 < \alpha < 2$ the statements

1) $\|f(x) - (G_n\, f)(x)\|_p = O\left(\left(n^{-1/2}\, \frac{\varphi(x)}{\emptyset(x)}\right)^{\alpha}\right)$
2) $\omega_2^{\emptyset}(f,\delta)_P = O(\delta^{\alpha})$, $\delta > 0$,

are equivalent.
Proof
Using Theorem 3.3.4, we obtain

$$\left\|(G^{"}{}_n f)(x)\right\|_p \leq c(p)\frac{n}{\emptyset^2(x)}\, \|f\|_p.$$

By Theorem 3.3.4 and Proposition 3.3.6 we get
1)Implies (2) and (2) implies (1)

Conclusion

for a function $f \in L_{p[0,1]}$,$0 < p < 1,$ we can find a positive and linear approximation if f is positive and linear.

References

1. M. Becker, Global approximation theorems for Sasz-Mirakjan and Baskakov operators in polynomial weighted spaces, Indiana Univ. Math. J. 27 (1978), 127-142.
2. H. Berens and G. G. Lorentz, Inverse theorems for Bernstein polynomials, Indiana Univ. Math. J. 21 (1972), 693-708
3. R. A. De Vore and G. G. Lorentz, ``Constructive Approximation,'' Springer-Verlag, Berlin /New York, 1993.
4. Z. Ditzian, Rate of approximation of linear processes, Acta Sci. Math. 48 (1985), 103-128.
5. Z. Ditzian, Rate of convergence for Bernstein polynomials revisited, J. Approx. Theory 50 (1987), 40-48.
6. Z. Ditzian, Direct estimate for Bernstein polynomials, J. Approx. Theory 79 (1994),165-166.

7. Z. Ditzian and D. Jiang, Approximation of functions by polynomials in C(&1, 1), Canad. J. Math. 44 (1992), 924-940.
8. Z. Ditzian, D. Jiang, and D. Leviatan, Simultaneous polynomial approximation, SIAM J. Math. Anal. 24 (1993), 1652-1661.
9. Z. Ditzian and V. Totik, ``Moduli of Smoothness,'' Springer-Verlag, New York, 1987.
10. M. Felten, Direct and inverse estimates for Bernstein polynomials, Constr. Approx. 14 (1998), 459-468.
11. I. Gavrea and D. H. Mache, Generalization of Bernstein-type approximation methods, in ``Approximation Theory, Proceedings, IDoMat 95'' (M. W. Mu ller, M.
12. Felten, and D. H. Mache, Eds.), Mathematical Research, Vol. 86, pp. 115-126, Akademie Verlag, Berlin, 1995
13. M. E. H. Ismail and C. P. May, On a family of approximation operators, J. Math. Anal. Appl. 63 (1978), 446-462.
14. D. H. Mache and D. X. Zhou, Characterization theorems for the approximation by a family of operators, J. Approx. Theory 84 (1996), 145-161.
15. G. Mastroianni and P. Ve rtesi, Approximation by some operators using a certain new modulus of smoothness, in ``Proceedings Second International Conference in Functional Analysis and Approximation Theory, Acquafredda di Maratea, 1992,'' pp. 387-397.
16. K. Sato^, Global approximation theorems for some exponential-type operators, J. Approx.Theory 32 (1981), 32-46.
17. D.-X. Zhou, On a conjecture of Z. Ditzian, J. Approx. Theory 69 (1992), 167-172.
18. Rinartha, K., Suryasa, W., & Kartika, L. G. S. (2018). Comparative Analysis of String Similarity on Dynamic Query Suggestions. In 2018 Electrical Power, Electronics, Communications, Controls and Informatics Seminar (EECCIS) (pp. 399-404). IEEE.
19. Suryasa, I. W., Rodríguez-Gámez, M., & Koldoris, T. (2021). Get vaccinated when it is your turn and follow the local guidelines. International Journal of Health Sciences, 5(3), x-xv. https://doi.org/10.53730/ijhs.v5n3.2938
20. Djuraev, A. M., Alpisbaev, K. S., & Tapilov, E. A. (2021). The choice of surgical tactics for the treatment of children with destructive pathological dislocation of the hip after hematogenous osteomyelitis. International Journal of Health & Medical Sciences, 5(1), 15-20. https://doi.org/10.21744/ijhms.v5n1.1813