Saturation problems for positive linear approximation of function in quasi normed spaces

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Abstract---Many authors work on constrained approximation such as monotonicity, convexity and k-monotonicity, but little works introduced in positive collinear approximation. The aim of our paper is to investigate linear and positive approximation for real functions in $L_{\infty}[0,1]$; saturation problem between degree of best positive and linear approximation, and find collinear positive best approximation for measurable function in $L_{\infty}[0,1]$ Lebesgue quasi normed spaces.

Keywords---saturation, positive, approximation, quasi normed spaces.

Introduction

In [6] the authors proved the following direct estimate for Bernstein operator:

$$|f(x) - (B_n f)(x)| \leq C \omega_{\varphi}^z \left( f, n^{-\frac{1}{2}}, \varphi(x)^{1-\lambda} \right) \quad x \in I = [0,1]$$

(1)

in which $\varphi: [0,1] \to \mathbb{R}$ is an admissible step weight function for details about $\varphi$ see [9].

If $\lambda = 0$ in (3.1) we get classical local estimate while when $\lambda = 1$ we get global norm estimate developed by Ditzian and Totik. So (3.1) fill the gap between the local and global approximation theorems for the Bernstein operator. Such result for polynomial approximation for details see [7,8 and 14]

Inequality (3.1) show that the error $f(x) - (B_n f)(x)$ is bounded pointwise by
\[ C \left( n^{-\frac{1}{2}} \varphi(x)^{1-\lambda} \right)^\alpha \] if \( \omega_{\varphi,\delta}(f, \delta) = O(\delta^\alpha) \) and \( \alpha \in [0,2] \).

In [5,16] the authors proved the converse result also true. \( \omega_{\varphi,\delta}(f, \delta) = O(\delta^\alpha) \) can be estimated in terms of the Bernstein operator that is the equivalence

\[ |f(x) - (B_n f)(x)| = O \left( \left( f, n^{-\frac{1}{2}}, \varphi(x)^{1-\lambda} \right)^\alpha \right) \iff \omega_{\varphi,\delta}(f, \delta) = O(\delta^\alpha) \]

Holds whenever \( \alpha \in (0,2) \) and \( \lambda \in [0,1] \).

We mean by \( g = O(f) \) that \( g \leq c(f) \) for some constant \( C \).

In [10] (1) can be considered as a further estimate for the general estimate

\[ |f(x) - (B_n f)(x)| \leq C \omega_{\varphi,\delta} \left( f, n^{-\frac{1}{2}}, \frac{\varphi(x)}{\varphi(\lambda)} \right) \] (2)

\( x \in [0,1] \) where \( \varphi : [0,1] \to \mathbb{R} \) is an admissible step_weight function of the Ditzian_Totik modulus [9] and \( \varphi^2 \) is a concave function.

(2) improve meet of (1) if \( \emptyset \) is replaced by \( \varphi^2, \lambda \in [0,1] \).

In [10] the authors proved inverse result to (2) for \( x \in (0,2) \). i.e.

\[ |f(x) - (B_n f)(x)| \leq C_1 \left( n^{-\frac{1}{2}} \frac{\varphi(x)}{\varphi(\lambda)} \right)^\alpha, \quad x \in [0,1] \quad n = 1,2,3, \ldots \]

Implies \( \omega_{\varphi,\delta}^2(f, \delta) \leq C_2(\delta^\alpha) \) if, in addition \( \varphi^2/\emptyset^2 \) is concave which is satisfied for \( \emptyset = \varphi^\lambda, \lambda \in [0,1] \) in particular.

**Auxiliary Results**

Here we introduce our auxiliary results that we need in our proof. We can modify a result in [15] to obtain the following Lemma.

In [15], the author used \( g_n \) as a positive function with integral equal to 1 and \( \emptyset \) is any weight function.

**Lemma 2.1** [15]

\[ \int_0^1 \frac{d^2}{dx^2} g_n(x, u) \leq c(p) \left( 4 + \frac{\emptyset^2(x)}{n} \right) \]

\[ + 2 \left( \frac{n^2}{\emptyset^4(x)} \int_0^1 \frac{\emptyset^2(x)}{n} - (u - x) (u - x)^3 g_n(x, u) du \right) \leq M \]

and \( M \) must be an absolute constant independent of \( n \) and \( x \).
Lemma 2.2 [15]

\[ \left| \int_x^t g^\prime(s)(t - s) \, ds \right| \leq \| \phi^2 g^\prime \|_p \, \frac{(t - x)^2}{\phi^2(x)} \]

Lemma 2.3 [9]

For \( f \in L_p(J) \) \( 0 < p \leq \infty \) we have

\[ \omega^2_\phi(f, \delta)_p \approx K^2_\phi(f, \delta^2)_p \]

Lemma 2.4[6]

\[ \Delta^2_h(f)(x) = \int_{h/2}^{h} \int_{h/2}^{h} f^\prime(x + s + t) \, ds \, dt \]

**The Main Results**

In this paper we prove direct estimates for the approximation of function in \( L_p \), \( 0 < p < 1 \) space using positive linear operator. and for pointwise linear operator. Also, we estimate inverse result for positive linear approximation.

Define

\[ w_\phi(f, \delta)_p = \sup_{|h| \leq \delta} \| f(x + \phi(x)h - f(x - \phi(x)h) \|_{L_p(I)} \]

Where \( x + \phi(x)h, x + \phi(x)h \in I \)

\[ \tilde{\omega}_\phi(f, \delta)_p = \sup_{|h| \leq \delta} \| f(x + \phi(x)h - f(x) \|_{L_p(I)} \]

\( x, x + \phi(x)h \in I \)

For the equivalence of \( w_\phi(f, \delta)_p \) and \( \tilde{\omega}_\phi(f, \delta)_p \) see [D.T] [9]

Now let us prove our first main result:

**Theorem 3.1**

Let \( f \in L_p[0,1] \) and \( \phi: [0,1] \rightarrow R \) be step function let

\[ A_n: L_p[0,1] \rightarrow L_p[0,1], n \in N \]

be bounded positive linear operators so that

\[ \| (A_n f)^\prime(x) \|_p \leq c(p) \frac{n}{\phi^2(x)} \| f \|_p \], \hspace{1cm} (3) \]

for \( x \in [0,1] \), \( f \in L_p[0,1] \), and

\[ \| \phi^2(A_n g)^\prime \|_p \leq c(p) \| \phi^2 g^\prime \|_p \], \hspace{1cm} (4) \]

where \( \phi^2, \phi^2 \) and \( \phi^2/ \phi^2 \) are concave function on \([0,1]\) and \( \alpha \in (0,2) \). Then for \( f \in L_p[0,1] \), the pointwise approximation

\[ \| f - (A_n f)(x) \|_p \leq c(p) \left( n^{-2} \frac{\phi(x)}{\phi^2(x)} \right)^\alpha \] \hspace{1cm} n = 1,2,.. \hspace{1cm} (5) \]

implies

\[ \omega^2_\phi(f, \delta)_p \leq c(p)\delta^\alpha \hspace{1cm} \delta > 0. \]
Proof

Let \( x, h \in [0,1] \) so that \( x + h \in [0,1] \) and let

\[
(\Delta_h^2 f)(x) = f(x + h) - 2f(x) + f(x - h) .
\]

and

\[
\| (\Delta_h^2 f)(x) \|_p = \| \Delta_h^2 (f - A_n + A_n)(x) \|_p
\]

\[
= \| (\Delta_h^2 (f - A_n f))(x) + (\Delta_h^2 A_n f)(x) \|_p
\]

\[
\leq 2^\frac{1}{p} \left( \int_0^1 |(\Delta_h^2 (f - A_n f))(x)|^p \right)^{\frac{1}{p}} + 2^\frac{1}{p} \left( \int_0^1 |(\Delta_h^2 A_n f)(x)|^p \right)^{\frac{1}{p}} .
\]

Using Lemma 2.3 and

\[
\|(\Delta_h^2 (f - A_n f))(x)\|_p = \|(f - A_n)(x + h) - 2(f - A_n)(x) + (f - A_n)(x - h)\|_p
\]

\[
\leq 2^\frac{1}{p} \left( \|(f - A_n)(x + h)\|_p + 2\|(f - A_n)(x)\|_p + \|(f - A_n)(x - h)\|_p \right)
\]

Using (5) we obtain

\[
\| (\Delta_h^2 (f - A_n f))(x) \|_p \leq 2^\frac{1}{p} \left( \left( \frac{\varphi(x + h)}{\Theta(x + h)} \right)^\alpha + 2 \left( \frac{\varphi(x)}{\Theta(x)} \right)^\alpha + \left( \frac{\varphi(x - h)}{\Theta(x - h)} \right)^\alpha \right) n^{-\frac{\alpha}{2}}
\]

\[
\leq c(p)n^{-\frac{\alpha}{2}} \left( \frac{\varphi(x)}{\Theta(x)} \right) . \tag{7}
\]

Using Lemma 2.3 \( \omega_0^\alpha (f, \delta) \) and \( K_\varphi^\alpha (f, \delta) \) are equivalent a

\[
c_2 \ K_\varphi^\alpha (f, \delta) \leq \omega_0^\alpha (f, \delta) \leq c_1 \ K_\varphi^\alpha (f, \delta)
\]

where \( c_1 \) and \( c_2 \) are absolute constants.

\[
K_\varphi^\alpha (f, \delta) = \| f - g \|_p + \delta^2 \| \Theta g^* \|_p \leq c \omega_0^\alpha (f, \delta)
\]

\[
\| f - g \|_p \leq A \omega_0^\alpha (f, \delta) , \quad \| \Theta g^* \|_p \leq B \delta^{-2} \omega_0^\alpha (f, \delta) .
\]

From (3) and (4) we obtain

\[
\| (A_n f)^{\alpha}(y) \|_p \leq \| A_n \alpha (f - g)(y) + (A_n \alpha g)(y) \|_p
\]

\[
\leq 2^\frac{1}{p} \left( \int_0^1 |A_n \alpha (f - g)(y)|^p \right)^{\frac{1}{p}} + 2^\frac{1}{p} \left( \int_0^1 |(A_n \alpha g)(y)|^p \right)^{\frac{1}{p}}
\]

\[
\leq c(p) \frac{n}{\varphi^2(y)} \| f - g \|_p + c(p) \frac{1}{\Theta^2(y)} \| \Theta g^* \|_p
\]

\[
= c(p) \frac{n}{\varphi^2(y)} \| f - g \|_p + c(p) \frac{\delta^{-2} \delta^2}{\Theta^2(y)} \| \Theta g^* \|_p
\]

\[
\leq c(p) \left( \frac{n}{\varphi^2(y)} + \frac{1}{\Theta^2(y) \delta^2} \right) K_\varphi^\alpha (f, \delta)
\]

\[
= c(p) \left( \frac{n}{\varphi^2(y)} + \frac{1}{\Theta^2(y) \delta^2} \right) \omega_0^\alpha (f, \delta) \tag{8}
\]
Using Lemma 2.4, we obtain
\[ \| (\Delta^2_n A_n f)(x) \|_p = \left\| \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} (A_n^* f)(x + s + t) ds \ dt \right\|_p. \]

Since \( A_n \) is bounded operators
\[ = \left( \int_{-1}^{1} \int_{-1}^{1} (A_n^* f)(x + s + t) ds \ dt \right)^{\frac{1}{p}} \]
\[ \| (\Delta^2_n A_n f)(x) \|_p \leq c \left( \int_{-1}^{1} |h^2 A_n^* (f)(x)|^p \right)^{\frac{1}{p}}, \]

Then using (8) we obtain
\[ \| (\Delta^2_n A_n f)(x) \|_p \leq c(p) h^2 \left( \frac{n}{\varphi^2(x)} + \frac{1}{\delta^2 \phi^2(x)} \right) \omega^2_\delta (f, \delta)_p. \]

Using (6) we get
\[ \| (\Delta^2_n A_n f)(x) \|_p \leq c(p) \left( \frac{\varphi(x)}{\phi(x)} \right)^{\alpha} n^{-\frac{\alpha}{2}} + \left( \frac{n h^2}{\varphi^2(x)} + \frac{h^2}{\delta^2 \phi^2(x)} \right) \omega^2_\delta (f, \delta)_p. \]

This implies
\[ \omega^2_\delta (f, \delta)_p \leq c(p) \left( \delta^{\alpha} + \delta + \frac{h^2}{\delta^2} \right) \]
\[ \leq c(p) \delta^{\alpha}. \]

**Notations and Defines**

Define the sequence of operators:
\[ Q_n (f(x)) = \int_{a}^{b} g_n(u,x) f(u) du \quad , a, b \in \mathbb{R}, \quad n \in \mathbb{N} \quad (9) \]

Such that
\( g_n(u,x) \) is positive for any \( x \in [a, b] \)
and
\[ \int_{a}^{b} g_n(u,x) \ du = 1 \quad \text{for any} \quad n \in \mathbb{N} \]

Also, we need to define
\[ \frac{d}{dx} g_n(u,x) = \frac{n}{\varphi^2} g_n(u,x) (u - x) \quad , n \in \mathbb{N} \]
where $\emptyset$ is a step weight function.

we use $L^2_{p[0,1]} = \{f : [a,b] \to R : f, f' \in L^p_{p[0,1]}\}$ which is called 2-fold $L_p$ space

**Proposition 3.2**

Let $g \in L^2_{p[0,1]}$, then for every function $\emptyset : [0,1] \to R$ there exists $M > 0$
\[
\|\emptyset^2 (G_n g)^\ast\|_p \leq M \|\emptyset^2 g^\ast\|_p, \quad n \in \mathbb{N}
\]

Where $M$ is an absolute constant.

Proof

By Taylor's formula we have
\[
(G_n g)^\ast = \int_0^1 \left[ \frac{d^2}{dx^2} g_n(u,x) \right] (g(x) + g'(x)(u - x)) + \int_x^u (u - s) g'(x) ds \ du
\]
\[
= \int_0^1 \left[ \frac{d^2}{dx^2} g_n(u,x) \right] (g(x) + g'(x)(u - x)) du + \int_0^1 \left[ \frac{d^2}{dx^2} g_n(u,x) \right] \left( \int_x^u (u - s) g'(x) ds \right) du
\]

Because
\[
\int_0^1 \left[ \frac{d^2}{dx^2} g_n(u,x) \right] (u - x)^i du = 0 \quad \text{for } i = 0, 1
\]

we obtain
\[
\|\emptyset^2 (G_n g)^\ast\|_p = \int_0^1 \left[ \frac{d^2}{dx^2} g_n(u,x) \right] \int_x^u (u - s) g'(x) ds \ du
\]
\[
= \int_0^1 \left[ \frac{d^2}{dx^2} g_n(u,x) \right] \left( \int_x^u (u - s) g'(x) ds \right) du
\]
\[
\leq \left( \int_0^1 \left| \emptyset^2 (x) \right| \int_0^1 \left| \frac{d^2}{dx^2} g_n(u,x) \right| \left( \int_x^u (u - s) g'(x) ds \right) du \right)^p dx^{1/p}
\]

By using Lemma 2.2 we get
\[
\|\emptyset^2 (G_n g)^\ast\|_p \leq \left( \int_0^1 \left| \emptyset^2 (x) \right| \int_0^1 \left| \frac{d^2}{dx^2} g_n(u,x) \right| \left( \emptyset^2 (x) g'(x) \frac{(u - s)^2}{\emptyset^2 (x)} \right) du \right)^p dx^{1/p}
\]
\[
\leq \left( \int_0^1 \left| \emptyset^2 (x) g'(x) \right| \int_0^1 \left| \frac{d^2}{dx^2} g_n(u,x) \right| \left( (u - s)^2 \right) du \right)^p dx^{1/p}
\]

Using Lemma 2.1 and boundedness of $g'$, we obtain
\[ \|\varphi^2(G_n g)\|_p \leq M \left( \int_0^1 (|\varphi^2(x)g'(x)|^p dx)^{1/p} \right) \]

\[ = M\|\varphi^2 g'\|_p \]

**Theorem 3.3**

Let \( G_n, n \in N \), be exponential type operator

\[ (G_n f) = \int_0^1 g_n(u,x)f(u)du, \quad n \in N \]

If \( \varphi^2 \) and \( \frac{\varphi^2}{\varphi^2} \) are concave function on \([0,1]\) then for \( f \in L_p[0,1] \) and \( 0 < \alpha < 2 \) the statements

1) \( \|f(x) - (G_n f)(x)\|_p = O\left( n^{-1/2} \frac{\varphi(x)}{\varphi(x)}^{\alpha} \right) \)

2) \( \omega_2^p(f, \delta) = O(\delta^{2\alpha}) \), \( \delta > 0 \),

are equivalent.

Proof

Using Theorem 3.3.4, we obtain

\[ \|(G_n' f)(x)\|_p \leq c(n) \frac{n}{\varphi^2(x)} \|f\|_p. \]

By Theorem 3.3.4 and Proposition 3.3.6 we get

1) Implies (2) and (2) implies (1)

**Conclusion**

for a function \( f \in L_p[0,1] \), \( 0 < p < 1 \), we can find a positive and linear approximation if \( f \) is positive and linear.

**References**

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