How to Cite:
Nair, P. C. P., & Baby, T. A. (2022). 2-domination polynomials of graphs. *International Journal of Health Sciences*, *6*(S6), 875–882. https://doi.org/10.53730/ijhs.v6nS6.9708

2-domination polynomials of graphs

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***Abstract*---**Let G be a simple graph of order m. Let D2(G, i) be the family of 2-dominating sets in G with size i. The polynomial D2(G, x) = ∑_(i=γ_2 (G))^m▒〖d_2 (G,i)x^i 〗 is called the 2-domination polynomial of G. Let D2(Sm, i) be the family of 2-dominating sets of the spider graph Sm with cardinality i and let d2(Sm, i) = |D2(Sm, i)|. Then the 2-domination polynomial D2(Sm, x) of Sm is defined as D2(Sm, x) = ∑_(i=γ_2 (S_m))^m▒〖d_2 (S_m,i)x^i 〗, where γ_2 (S_m) is the 2-domination number of Sm. In this paper, we obtain some operations on graphs.

***Keywords*---**2-dominating set, 2-domination number, 2-domination polynomial.

Introduction

Let G = (V, E) be a simple graph of order m. For any vertex v∈V, the open neighbourhood of V is the set N(v) = {u∈V / uv∈E} and the closed neighbourhood of V is the set N[v] = N(v) ∪ {v}. For a set S ⊆V, the open neighbourhood of S is N(S) = $\cup_{v\in S} N(v)$ and the closed neighbourhood of S is N[S] = N(S) ∪ S. A set D ⊆ V is a dominating set of G if N[D] = V or equivalently, every vertex in V − D is adjacent to at least one vertex in D. The domination number of the graph G is defined as the minimum size taken over all dominating sets D of vertices in G and is denoted by γ(G). A set D ⊆ V is a 2-dominating set if every vertex in V−D is adjacent to at least two vertices in D. The 2-domination number of a graph G is defined as the

Manuscript submitted: 9 March 2022, Manuscript revised: 27 May 2022, Accepted for publication: 18 June 2022

minimum size taken over all 2-dominating sets of vertices in G and is denoted by γ_2(G).

2-Domination Polynomials Of Graph Operations

Definition 2.1

Let G be a simple graph of order m. A subset D ⊆ V is a 2-dominating set of the graph G, if every vertex $v \in$ V–D is adjacent to at least 2 vertices in D. The 2-domination number γ_2(G) is the minimum cardinality among the 2-dominating sets of G.

Definition 2.2

Let D_2(G, i) be the family of 2-dominating sets in G with cardinality i. The polynomial

$D_2(G, x) = \sum_{i=\gamma_2(G)}^{|V(G)|} d_2(G,i)x^i$ is called the 2-domination polynomial of G. where $\gamma_2(G)$ is the 2-domination number of G.

Example 2.3

Consider the graph G given in Figure 2.1

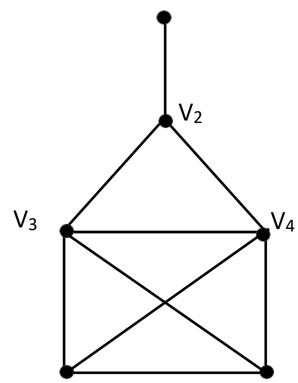


Figure 2.1

There is no 2-dominating sets of G of cardinality 1 and cardinality 2.

The 2-dominating sets of G of cardinality 3 are {{v1, v3, v4}, {v1, v3, v5},

{v1, v3, v6} {v1, v4, v5}, {v1, v4, v6}}.

Therefore, d2(G, 3) = 5.

The 2-dominating sets of G of cardinality 4 are {{v1, v2, v3, v4}, {v1, v2, v3, v5},

{v1, v2, v3, v6},{v1, v2, v4, v5}, {v1, v2, v4, v6}, {v1, v2, v5, v6}, {v1, v3, v4, v5}, {v1, v3,v4, v6}, {v1, v3, v5, v6}, {v1, v4, v5, v6}}.
Therefore, d2(G, 4) = 10.

The 2-dominating sets of G of cardinality 5 are {{v1, v2, v3, v4, v5}, {v1, v2, v3,v4, v6}, {v1, v2, v3, v5, v6}, {v1, v2, v4, v5, v6}, {v1, v3, v4, v5, v6}}.
Therefore, d2(G, 5) = 5.

The 2-dominating set of G of cardinality 6 is {v1, v2, v3, v4, v5, v6}. Therefore, d2(G, 6) = 1.

Since, the minimum cardinality is 3, $\gamma_2(G)$ = 3.
Therefore, the 2-domination polynomial of G is

$D_2(G, x) = \sum_{i=\gamma_2(G)}^{m} d_2(G, i)x^i.$

$= \sum_{i=3}^{6} d_2(G, i)x^i.$

$= d_2(G, 3)x^3 + d_2(G, 4)x^4 + d_2(G, 5)x^5 + d_2(G, 6)x^6.$

$= 5x^3 + 10x^4 + 5x^5 + x^6.$

$D_2(G, x) = x(1 + x)^5 - (x + 5x^2 + 5x^3).$

Theorem 2.4

Let K_m be a complete graph of order m. Then $D_2(K_m \circ \overline{K_n}, x) = x^{mn}(1+x)^m$, for all $m, n \geq 2$.

Proof

Since K_m has m vertices, $K_m \circ \overline{K_n}$ has m(n+1) vertices. Clearly,$\{v_{m+1}, v_{m+2}, v_{m+3}, \ldots, v_{mn+m}\}$ is the minimum 2-dominating set of $K_m \circ \overline{K_n}$.
Therefore,$\gamma_2(K_m \circ \overline{K_n})$ = mn
Obviously, there are $\binom{m}{i}$ chances for 2-dominating sets of $K_m \circ \overline{K_n}$ with cardinality mn+i
where $1 \leq i \leq mn$.

Here, $D_2(K_m \circ \overline{K_n}, x) = x^{mn} + \binom{m}{1}x^{mn+1} + \binom{m}{2}x^{mn+2} + \ldots + x^{mn+m}$

$= x^{mn}[1 + \binom{m}{1}x + \binom{m}{2}x^2 + \ldots + x^m]$

$= x^{mn}[1 + x]^m$, for all $m, n \geq 2$.

Example 2.5

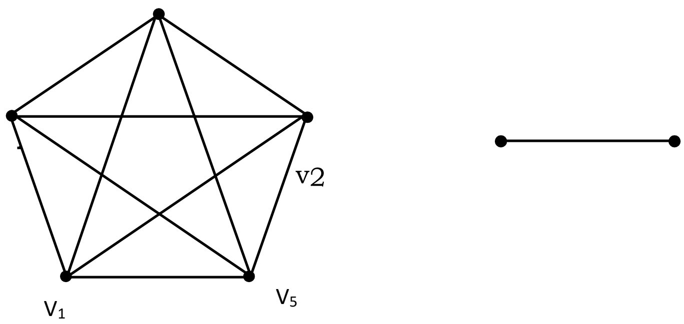
Consider $K_5 \circ \overline{K_2}$ be the graph given in Figure 2.2.

V3

K5:

V2

V4



v11 V10

K5 ∘ $\overline{K_2}$:

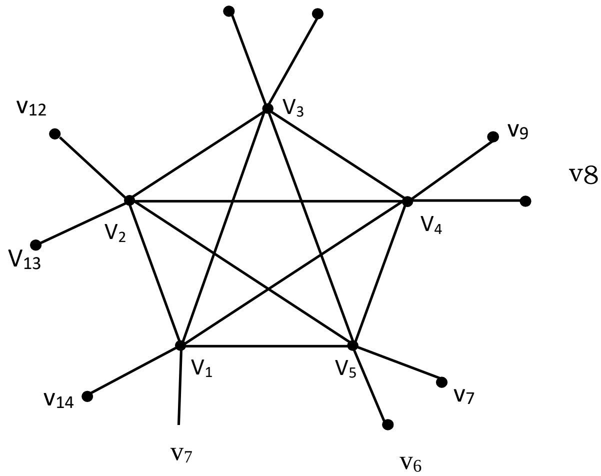


Figure 2.2

There is no 2-dominating sets of $K_5 \circ \overline{K_2}$ of cardinality 1, 2, 3, 4, 5, 6, 7, 8, 9.
The minimum cardinality of $K_5 \circ \overline{K_2}$ is 10.
Therefore, $\gamma_2(K_5 \circ \overline{K_2}) = 10$
The 2-dominating set of $K_5 \circ \overline{K_2}$ with cardinality 10, 11, 12, 13, 14, 15.

$d_2(K_5 \circ \overline{K_2}, 10) = \binom{5}{0} = 1.$

$d_2(K_5 \circ \overline{K_2}, 11) = \binom{5}{1} = 5.$

$d_2(K_5 \circ \overline{K_2}, 12) = \binom{5}{2} = 10.$

$d_2(K_5 \circ \overline{K_2}, 13) = \binom{5}{3} = 10.$

$d_2(K_5 \circ \overline{K_2}, 14) = \binom{5}{4} = 5.$

$d_2(K_5 \circ \overline{K_2}, 15) = \binom{5}{5} = 1.$

Therefore, $D_2(K_5 \circ \overline{K_2}, x) = \sum_{i=10}^{15} d_2(K_5 \circ \overline{K_2}) x^i$
$= x^{10} + 5x^{11} + 10x^{12} + 10x^{13} + 5x^{14} + x^{15}$

$$= x^{10}\,(1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5)$$
$$= x^{10}(1 + x)^5.$$

Corollary 2.6

Let K_m be a complete graph of order m. Then, $D_2(K_m \circ \overline{K_m}, x) = x^{m^2}(1+x)^m$, for all $m \geq 2$.

Theorem 2.7

The 2-domination polynomial of $K_m \vee P_2$ is $D_2(K_m \vee P_2, x) = (1 + x)^{m+2} - [1 + (m+2)x]$.

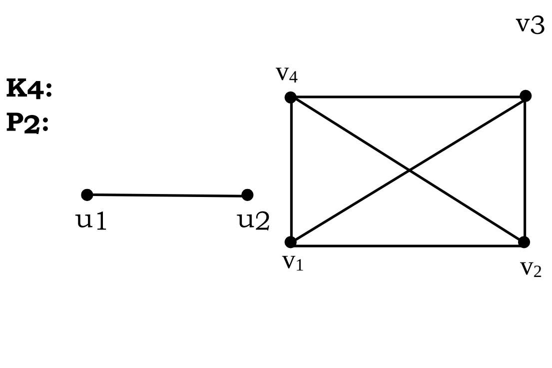
Proof

Let K_m be a complete graph with order m and P_2 be the path with order 2.
Label the vertices of $K_m \vee P_2$ as, $v_1, v_2, v_3, \ldots, v_m, u_1, u_2$.
Since K_m has m vertices, $K_m \vee P_2$ has m+2 vertices.
The minimum cardinality of $K_m \vee P_2$ is 2.
Therefore, $\gamma_2(K_m \vee P_2) = 2$.
There are $\binom{m+2}{i}$ possibilities of 2-dominating sets of $K_m \vee P_2$ with cardinality m+2+i.

Hence,
$$D_2(K_m \vee P_2, x) = \sum_{i=\gamma_2(K_m \vee P_2)}^{|V(K_m \vee P_2)|} d_2(K_m \vee P_2, i)x^i$$
$$= \sum_{i=2}^{m+2} d_2(K_m \vee P_2, i)x^i$$
$$= \binom{m+2}{2}x^2 + \binom{m+2}{3}x^3 + \binom{m+2}{4}x^4 + \cdots + \binom{m+2}{m+1}x^{m+1} + \binom{m+2}{m+2}x^{m+2}$$
$$= \left[\sum_{i=0}^{m+2}\binom{m+2}{i}x^i\right] - 1 - (m+2)\,x$$
$$= (1+x)^{m+2} - 1 - (m+2)x$$
$$= (1+x)^{m+2} - [1 + (m+2)x]$$

Example 2.8

Consider $K_4 \vee P_2$ given in the Figure 2.3



V4

K4∨ P2: $_1$ V

u1 u2

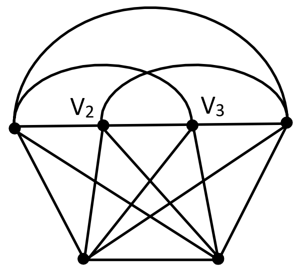


Figure 2.3

There is no 2-dominating set K4 ∨ P2 of cardinality 1.The minimum cardinality of K4 ∨ P2 is 2.

Therefore, $\gamma_2(K_4 \vee P_2) = 2$.

The 2-dominating set of K4 ∨ P2 with cardinality 2, 3, 4, 5, 6 are

$d_2(K_4 \vee P_2, 2) = \binom{6}{2} = 15$

$d_2(K_4 \vee P_2, 3) = \binom{6}{3} = 20$

$d_2(K_4 \vee P_2, 4) = \binom{6}{4} = 15$

$d_2(K_4 \vee P_2, 5) = \binom{6}{5} = 6$

$d_2(K_4 \vee P_2, 6) = \binom{6}{6} = 1$

Therefore, 2-domination polynomial of $K_4 \vee P_2$

$$D_2(K_4 \vee P_2, x) = \sum_{i=2}^{6} d_2(K_4 \vee P_2, i)x^i$$
$$= 15\,x^2 + 20\,x^3 + 15\,x^4 + 6x^5 + x^6$$
$$= (1 + x)^6 - 1 - 6x.$$

Theorem 2.9

The 2-domination polynomial of $P_2[K_m]$ is $D_2(P_2[K_m], x) = (1 + x)^{2m} - (1 + 2mx)$.

Proof

Let P_2 be the path with order 2 and K_m be the complete graph with order m.

Then, $P_2[K_m]$ has 2m vertices.

The minimum cardinality of $P_2[K_m]$ is $\gamma_2(P_2[K_m]) = 2$.

There are $\binom{2m}{i}$ possibilities of 2-dominating sets of $P_2[K_m]$ with cardinality i.

Hence, $D_2(P_2[K_m], x) = \sum_{i=\gamma_2(P_2[K_m])}^{|V(P_2[K_m])|} d_2(P_2[K_m], i)x^i$

$= \sum_{i=2}^{2m} d_2(P_2[K_m], i)x^i$

$$=\binom{2m}{2}x^2 + \binom{2m}{3}x^3 + \binom{2m}{4}x^4 + \cdots + \binom{2m}{2m-1}x^{2m-1} + \binom{2m}{2m}x^{2m}.$$

$$= \left[\sum_{i=0}^{2m}\binom{2m}{i}x^{i}\right] - 1 - 2mx$$

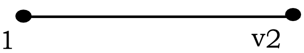
$$= (1+x)^{2m} - (1 + 2mx).$$

Example 2.10

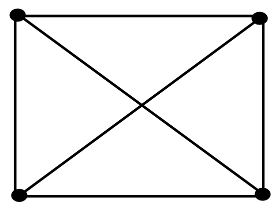
Consider $P_2[K_4]$ given in the figure 2.4.

u4 u3

P2:



K4:



u1 u2

1

P2[K4]:
(v1, u4)

5

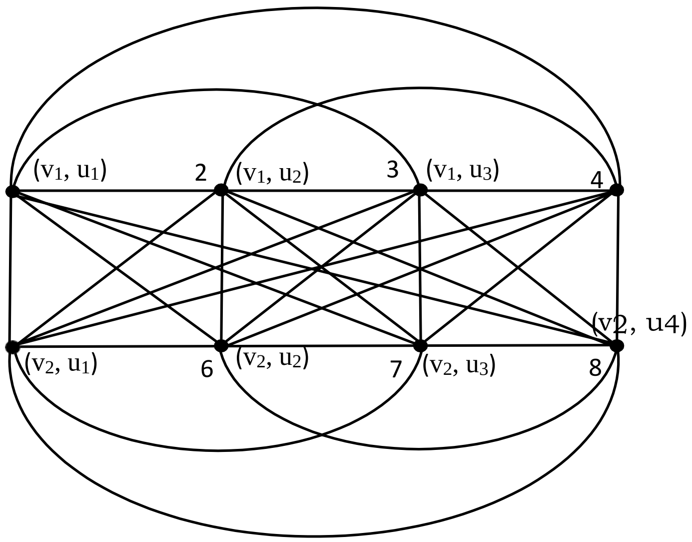


Figure 2.4

There is no 2-dominating sets of P2[K4] of cardinality 1.
The minimum cardinality of P2[K4] is 2.
Therefore, γ_2(P2[K4]) = 2.
The 2-dominating set of P2[K4] with cardinality 2, 3, 4, 5, 6, 7, 8 are,

$d_2(P_2[K_4], 2) = \binom{8}{2} = 28.$

$d_2(P_2[K_4], 3) = \binom{8}{3} = 56.$

$d_2(P_2[K_4], 4) = \binom{8}{4} = 70.$

$d_2(P_2[K_4], 5) = \binom{8}{5} = 56.$

$d_2(P_2[K_4], 6) = \binom{8}{6} = 28.$

$d_2(P_2[K_4], 7) = \binom{8}{7} = 8.$

$d_2(P_2[K_4], 8) = \binom{8}{8} = 1.$

Therefore, $D_2(P_2[K_4], x) = \sum_{i=2}^{8} d_2(P_2[K_4], i)x^i$

$= 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8.$

$= (1+x)^8 - (1+8x).$

Hence, $D_2(P_2[K_4], x) = (1+x)^8 - (1+8x).$

Conclusion

In this paper, 2-domination polynomials of graphs has been derived by identifying its 2-dominating sets. It also help us to characterize the 2-dominating sets of cardinality i. We can generalize this study to any graph and some interesting operation can be obtained via the roots of the 2-domination polynomial of graphs.

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