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Almost self-injective semimodules

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***Abstract*---**A New generalization of injective semimodule has been presented in this working paper. An Ȑ-semimodule Ɱ is called almost self-injective, if Ɱ is almost Ɱ-injective semimodule. Some properties of this notion have been presented. The relationship of this concept to some concepts has also been clarified as End (Ȑ) of indecomposable almost self-injective semimodules, Rad (Ȑ) also some related notions of it have been studied.

***Keywords*---**Semirings, semimodules, injective semimodules, almost-injective semimodules, almost self-injective semimodules.

Introduction

The topic semimodule is the most popular among researchers in the domain of algebra today as it is extension of modules. For this search another generalization of injective semimodules was taken which is almost self-injective semimodule. A semimodule Ɱ is said to be almost self-injective, if Ɱ is almost Ɱ-injective semimodule. Every self-injective is almost self-injective but the converse is not true. Any fully invariant direct summand of almost self- injective semimodule is also almost self- injective. Some concepts like, uniform, Rad (Ȑ)(Jacobson radical of semiring Ȑ), Z(Ɱ)(singular subsemimodule) related with our topic have been studied. A semiring Ȑ is a nonempty set with two operations addition and multiplication such that, Ȑ with first operation is commutative monoid with identity, Ȑ with second operation is monoid with identity different from the identity of the addition, the second operation distributes over the first and the identity element of the first operation is absorbing element of the second operation. The semiring Ȑ is commutative if the second operation is commutative. A semimodule Ɱ over semiring Ȑ can be defined similar to module over ring. A nonempty subset Ӿ of Ɱ is called subsemimodule of Ɱ, if Ӿ is closed under addition and scalar multiplication. For two Ȑ-semimodules Ɱ and Ɲ a homomorphism map can be defined as follow, ɸ:Ɱ⟶Ɲ is a homomorphism of Ȑ-

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semimodules if, ϕ(ȿ+ŧ) = ϕ(ȿ)+ϕ(ŧ); ϕ(ɼȿ) = ɼ ϕ(ȿ), ∀ȿ,ŧ∈ Ɱ and ɼ ∈ Ȓ, the set of all Ȓ-homomorphisms from Ɱ into Ɲ is denoted by Hom(Ɱ, Ɲ) and the set of endomorphisms (homomorphism from Ɱ into itself) is denoted by End(Ɱ) which is a semiring with respect to addition and multiplication defined as follow: $(\alpha+\beta)(ŧ) = \alpha(ŧ) + \beta(ŧ)$ and $\alpha^\circ\beta(ŧ) = \alpha(\beta(ŧ))$ where ŧ∈ Ɱ, , $\alpha, \beta \in$ End(Ɱ)[1]. If Ɱ is a left Ȓ-semimodule, then Ɱ can be made right ʆ-semimodule where ʆ= End(Ɱ)[2]. In this work, all semirings are assumed to be abelian with nonzero identity and all semimodules are unitary Ȓ-semimodule.

1. Some definitions and remarks.

It is necessary to know the basics that have been relied upon in our work, so they were clarified in this part of the work.

Definition1.1[3]. A semiring Ȓ is said to be semi-domain if ŧ ɼ = 0, implies either ŧ = 0 or ɼ = 0 for ŧ, ɼ in Ȓ.

Definition1.2[1]. Let Ɱ be a semimodule a subsemimodule Ӿ of Ɱ is said to be subtractive if,∀ȿ ,ŧ ∈ Ɱ and ȿ, ȿ+ŧ ∈Ӿ imply that ŧ ∈Ӿ, and denoted by Ӿ≤ Ɱ.

Definitin1.3[1]. A semimodule Ɱ is called semisubtractive if, ∀ȿ, ŧ ∈ Ɱ, there is ë ∈ Ɱ such that either ë +ŧ = ȿ or ȿ + ë = ŧ.

Definition1.4[1]. An element ȿ ∈ Ɱ is called cancellable if satisfies, ȿ +ŧ = ȿ +u implies u= ŧ. A semimodule Ɱ is said to be cancellative if every element of Ɱ is cancellable

Definition1.5[2]. An Ȓ-semimodule Ɱ is called $\mathcal{B}-$injective, if for each subsemimodule Ӿ of $\mathcal{B}$ and homomorphism δ:Ӿ⟶Ɱ can be extended to homomorphism $\xi: \mathcal{B} \longrightarrow$ Ɱ. A semimodule is called injective if it is injective relative to every semimodule. A semimodule Ɱ is called quasi-injective (self-injective in some literatures), if it is Ɱ-injective.

In [4], a generalization of injective semimodule is represented which is almost injective-semimodule as following.

Definition1.6[4]. If we have two semimodules Ɱ and $\mathcal{B}$, we say that Ɱ is almost $\mathcal{B}$ -injective semimodule if, for each subsemimodule Ǻ of $\mathcal{B}$ and each homomorphism ξ : Ǻ→Ɱ, either there is a homomorphism γ: $\mathcal{B}$ →Ɱ such that $\gamma\, i=\xi$. Or there is a homomorphism ∅:Ɱ → Y such that $\emptyset\xi = \pi i$,where Y is nonzero direct summand of $\mathcal{B}$ and $\pi: \mathcal{B} \to Y$ is projection map.

Definition1.7[5]. A nonzero semimodule Ɱ is called indecomposable, if the direct summands of it only 0 and Ɱ.

Definition1.8[5]. A semimodule *Ɱ* is called uniform if every subsemimodule of it is large in Ɱ.

Definition1.9[6]. The singular subsemimodule of *Ɱ* is defined by $\mathcal{Z}$(*Ɱ*) = {m ∈ Ɱ| $ann_{Ȓ}(m)$ is essential ideal in *Ȓ*}. If $\mathcal{Z}$(*Ɱ*) =*Ɱ*, then *Ɱ* is called singular semimodule and if $\mathcal{Z}$(*Ɱ*) = 0, *Ɱ* is called nonsingular.

Definition1.10[7]. The radical of semimodule *Ɱ* is the intersection of all maximal subsemimodules of *Ɱ*.

Definition1.11 [9]. A subsemimodule *Ӿ* of *Ɱ* is called fully invariant if for each β: Ɱ ⟶ Ɱ, then β(*Ӿ*) ⊆ *Ӿ*.

Definition1.12 [7]. A semiring is said to be local if it has only one maximal ideal.

Theorem1.13[7]. Let *Ȓ* be semiring , then the following statements equivalents:

(1) *Ȓ* is local.
(2) Rad(*Ȓ*) is maximal ideal of *Ȓ*.
(3) The set of all non -invertible elements of *Ȓ* is closed under addition.

(4) Rad($\grave{\grave{R}}$) = {ŧ ∈ Ȑ| ŧ is non- invertible element}.

Lemma 1.14[8]. For any element ɼ ∈ Ȑ, ɽ ∈ Rad(Ȑ) if and only if 1+ŧɼ is invertible for all ŧ ∈ Ȑ.

Definition 1.15[10]. A semi-field is a set Ƒ with two associative operation (+) and (×) satisfying the following conditions:

(1)(Ƒ, +) is a commutative monoid with identity element 0.

(2)(Ƒ/{0}, ×) is a group.

(3)The operation (×) distributives over (+).

(4)For every element ŧ∈ Ƒ, ŧ0 = 0ŧ = 0.

2. Almost self- injective semimodules

Definition 2.1. A semimodule Ɱ is said to be almost self-injective, if Ɱ is almost Ɱ- injective, this means either diagram (i) or diagram (ii) commutes.

$$\begin{tikzcd} Å \arrow[r, "i"] \arrow[d, "\xi"'] & Ɱ \arrow[dl, "\zeta"] \\ Ɱ & \end{tikzcd}$$

(i)

$$\begin{tikzcd} Å \arrow[r, "i"] \arrow[d, "\xi"'] & Ɱ = Ұ \oplus Ʒ \arrow[d, "\pi"] \\ Ɱ \arrow[r, "\gamma"'] & Ұ \end{tikzcd}$$

(ii)

A semiring Ȑ is said to be almost self-injective if $Ȑ_{Ȑ}$ is almost self-injective semimodule.

Remarks 2.2.

(1) If Ɱ is indecomposable, almost self-injective semimodule, then either Ɱ is quasi-injective or $\gamma\xi = 1_{Ɱ} i = i$, this means ξ is monomorphism.

(2) The following implications hold for any semimodule:

$$\begin{tikzcd} \text{Injective} \arrow[r] \arrow[d] & \text{Almost-injective} \arrow[r] & \text{Almost self-injective} \\ \text{Quasi-injective} \arrow[rr] & & {} \arrow[u] \end{tikzcd}$$

d Alhossaini defined semi-field of fraction of the semi-domain is in [3].

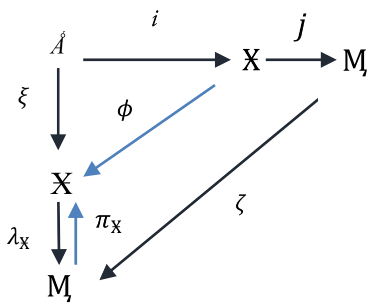
Definition 2.3. A valuation semi-domain is a semi-domain $\mathcal{D}$ such that for each element ŧ in semi-field of fraction Ƒ, either ŧ or $ŧ^{-1}$ belongs to $\mathcal{D}$.

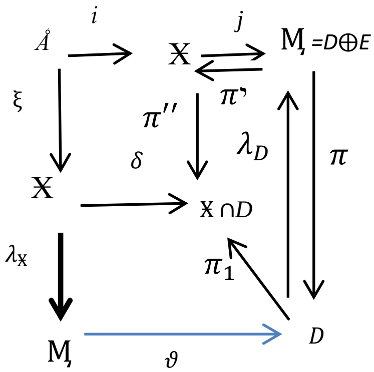
In [11], the following examples is discussed for modules, with respect to the semimodule the same result is reached.

Examples 2.4.

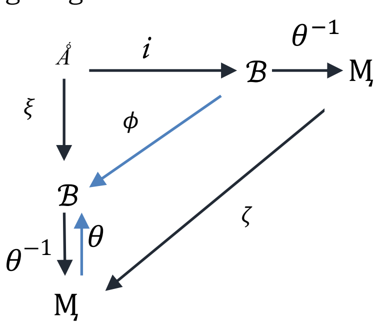
(1) Every injective semimodule is almost self-injective but the converse is not true, for instance $\mathbb{Z}_2$ as $\mathbb{N}$-semimodule is almost self-injective semimodule but not injective.

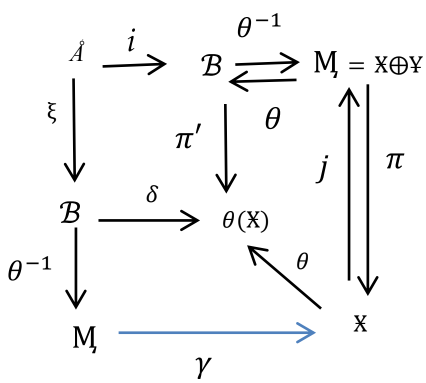
(2) Every self –injective semimodule is almost self-injective but the converse is not true, for example $\mathbb{N}/_{p} \oplus \mathbb{N}/_{p^2}$ as $\mathbb{N}$ semimodule where p is prime number is almost self-injective but it is not self- injective semimodule[11].
(3) Every valuation semi-domain which is not division semiring is almost self-injective but not self –injective semimodule[11].
Proposition 2.5. Any fully invariant summand of almost self-injective semimodule is also almost self-injective.
Proof: Suppose $Ӿ$ is a fully invariant direct summand of Ɱ, where Ɱ is almost injective-semimodule. Consider the following diagrams:





Where $Ǻ \leq$ Ӿ, since Ɱ is almost self-injective semimodule, then either there is endomorphism ζ of $Ɱ$ such that $\zeta j i = \lambda_{Ӿ}\ \xi$. Or there is ϑ: Ɱ $\longrightarrow D$ where D is nonzero direct summand of Ɱ such that $\vartheta\lambda_{Ӿ}\ \xi = \pi j i$. For the first diagram, define ϕ:Ӿ⟶Ӿ such that $\phi = \pi_{Ӿ}\zeta j$, then $\phi i = \pi_{Ӿ}\zeta j i = (\pi_{Ӿ}\lambda_{Ӿ})\ \xi = \xi$, as for the second diagram, since Ӿ is fully invariant, then $Ӿ$=(Ӿ ∩D) ⊕(Ӿ ∩E), this means Ӿ ∩ D is direct summand of Ӿ. Define δ: Ӿ ⟶ Ӿ ∩ D, such that $\delta = \pi''\pi'\lambda_D\vartheta\lambda_{Ӿ} = \pi_1\vartheta\lambda_{Ӿ}$ then $\delta\xi = \pi_1(\vartheta\lambda_{Ӿ}\xi) = \pi_1\ \pi j i = \pi'' i$. Hence $Ӿ$ is almost self-injective.
Proposition 2.6. Let Ɱ ≅ $\mathcal{B}$; Ɱ is almost self-injective semimodule if and only if $\mathcal{B}$ is almost self-injective.
Proof: Let θ: Ɱ ⟶ $\mathcal{B}$ be an isomorphism and Ɱ be almost-self-injective semimodule, then either there ∃ ζ ∈ End(Ɱ), such that $\zeta\theta^{-1}i = \theta^{-1}\xi$. Define $\emptyset: \mathcal{B} \to \mathcal{B}$ such that $\emptyset = \theta\ \zeta\theta^{-1}$, then $\emptyset i = \theta\zeta\theta^{-1} i = \theta\theta^{-1}\xi = \xi$. OR, there exists γ: Ɱ → ¥ where ¥ is nonzero direct summand of Ɱ such that $\gamma\ \theta^{-1}\xi = \pi\theta^{-1} i$. Define $\delta: \mathcal{B} = \theta$(Ӿ)⊕$\theta$(¥) ⟶ θ(Ӿ) where θ(Ӿ) is a nonzero direct summand of $\mathcal{B}$, such that $\delta = \theta\gamma\ \theta^{-1}$ we have $\delta\xi = \theta\gamma\theta^{-1}\xi = \theta\pi\theta^{-1} i = \pi' i$. Then $\mathcal{B}$ is almost self-injective. As in the following diagrams:





Similarly, if the assumption $\theta: \mathcal{B} \longrightarrow$ M and $\mathcal{B}$ is almost self-injective semimodule, then M is almost self-injective again. ∇

Remark 2.7 [5]. If a semimodule M is cancellative and semisubtractive, then M = D⊕E if and only if M = D+E and D∩ E=0.

Definition 2.8 [5]. A semimodule M is called π-injective if for any two subsemimodules D and E of M with D⋂ E= 0, there exist $\vartheta,\delta \in$ End(M) such that, ϑ and δ are idempotents; $\vartheta + \delta = 1_M$; D ⊆ker(ϑ) and E ⊆ ker(δ).

Remark 2.9. The definition (2.8) is equivalent to, a semimodule M is said to be π-injective semimodule if for any two subsemimodules Ӿ and ¥ of M with Ӿ∩¥ = 0, the projections π_i: Ӿ⊕ ¥ → Ӿ or ¥ can be lifted to an endomorphism of M. If M is π-injective and U_1, U_2 are subsemimodules with $U_1 \cap U_2=0$, then π_i: $U_1 \oplus U_2 \to U_i$ can be extended to idempotent endomorphism with sum equal to 1_M [5]. Conversely, if $U_1 \cap U_2=0$, where U_1, U_2 are subsemimodules of M and π_i: $U_1 \oplus U_2 \to U_i$ can be extended to idempotent endomorphism of M, then M is π-injective.

Proof: Since π_1: $U_1 \oplus U_2 \to U_1$ can be extended to ρ: M →M, then π_1(x+ y) = x = ρ(x)also π_2: $U_1 \oplus U_2 \to U_2$ can be extended to σ:M→M, then π_2(x+ y) = y = σ(y), where x in U_1 and y in $U_2 \Longrightarrow \rho|_{U_1} = \pi_1$ and $\sigma|_{U_2} = \pi_2$, ρ(x) +σ(x) = x, ∀x∈ U_1 (since $U_1 \subseteq$ ker(π_2) = ker(σ)), also ρ(y) +σ(y) = y, ∀y∈ U_2 ($U_2 \subseteq$ ker(π_1) = ker(ρ)) ⟹ ρ+σ = 1_M . Thus M is π-injective.∇

Recall, a semimodule M is said to be maximal essential extension of subsemimodule Ł if $\mathcal{N}$ is proper extension of M, then $\mathcal{N}$ is not essential extension of Ł. An Ȑ- semimodule $\mathcal{N}$ is said to be injective hull of semimodule M, if $\mathcal{N}$ is injective and it is essential extension of M, denoted by E(M) [5].

Note.[1]. It is well-known that every module over a ring has an injective hull, but this does not hold in general, for semimodules over a semiring.

Remark 2.10[5]. If M is π-injective semimodule having injective hull and E(M)= Ḓ⊕Ḝ, where Ḓ and Ḝ are subsemimodules of E(M), then M = (Ḓ ∩ M)⊕(Ḝ∩ M).

Remark 2.11 [5]. If M is π-injective, indecomposable semimodule having injective hull, then M is uniform.

Proposition 2.12. An indecomposable, almost self-injective semimodule M is π-injective.

Proof: Suppose U_1, $U_2 \neq 0$ are subsemimodules of indecomposable almost self-injective semimodule M with $U_1 \cap U_2=0$, the projection π: $U_1 \oplus U_2 \to U_i$ either can be extended to endomorphism of M, or there exists an Ȑ-homomorphism α of M such that απ = i ⟹ ker (π) = 0 which is contradiction (since ker π = $U_i \neq 0$ where i = 1, 2) hence M is π-injective.∇

Proposition 2.13. If M is An indecomposable, almost self-injective semimodule with injective hull E(M), then M is uniform.

Proof: From Proposition 2.12, we have M is π-injective semimodule and from Remark 2.11. We get M is uniform. ∇

Proposition 2.14.[4]. Let M and N be uniform semimodules having injective hulls E(M) and E(N) respectively, then M is almost N-injective semimodule if and only if for every $\vartheta \in$ Hom (E(N), E(M)), then either ϑ (N) ⊆ M or ϑ is isomorphism and ϑ^{-1} (M) ⊆N.

The following proposition has been proved for module in [11], it will be proved for semimodule after it has been converted to suit the characteristics of semimodules where two conditions were added, cancellative and semisubtractive.

Proposition 2.15. Let Ȑ be semiring has no nontrivial idempotent and it having injective hull E($_{Ȑ}$Ȑ) which is cancellative, semisubtractive, then Ȑ is left almost self-injective semimodule if and only if for all ë ∈ E($_{Ȑ}$Ȑ), either ë ∈ Ȑ or ∃r∈ Ȑ such that r ë = 1.

Proof: Suppose that Ȑ is a left almost self-injective, then $_{Ȑ}$Ȑ is uniform semimodule(because it is indecomposable almost self-injective by proposition(2.13)), let ë ∈ E($_{Ȑ}$Ȑ) and $\mathcal{L}_{ë}$: Ȑ⟶ E($_{Ȑ}$Ȑ) be the right multiplication homomorphism by ë. Then there is endomorphism ψ of E($_{Ȑ}$Ȑ) extension of $\mathcal{L}_{ë}$ by proposition (2.14) either ψ (Ȑ) ⊆ Ȑ or ψ is an isomorphism and ψ^{-1}(Ȑ) ⊆ Ȑ. If ψ(Ȑ) ⊆ Ȑ⟹ ë ∈ Ȑ. If ψ is isomorphism and ψ^{-1} (Ȑ) ⊆ Ȑ ⟹ $\psi^{-1}(1)$ ∈ Ȑ and there is ȓ ∈ Ȑ such that ψ(ȓ) = 1, so ȓ ë = $\mathcal{L}_{ë}$(ȓ) = ψ(ȓ) = 1. Conversely, suppose ∀ ë ∈ E($_{Ȑ}$Ȑ), then either ë ∈ Ȑ or ∃ȓ∈ Ȑ such that ȓë = 1. Claim E($_{Ȑ}$Ȑ) is uniform. If γ ∈ End (E($_{Ȑ}$Ȑ)) is idempotent, then either γ(1) ∈ Ȑ or there is ȓ ∈ Ȑ such that γ(1) r = 1. If γ(1) ∈ Ȑ, then γ(1) is an idempotent element in Ȑ by assumption either γ(1)= 0 or γ(1) = 1⟹ γ is zero or the identity map of E($_{Ȑ}$Ȑ) (because Ȑ $\leq_e$ E ($_{Ȑ}$Ȑ)). If r γ(1) =1 for some ȓ ∈ Ȑ, then γ(ȓ) =1, so γ(1) = γ(γ(ȓ)) = γ^2(ȓ) = γ(ȓ) = 1. Thus, $\gamma|_{Ȑ}$ = $1_{Ȑ}$⟹ γ = $1_{E(Ȑ)}$, otherwise, ∃ ȿ ∈ E($_{Ȑ}$Ȑ) such that γ(ȿ) ≠ȿ, by semisubtractive of E(Ȑ), there is ȶ ∈ E($_{Ȑ}$Ȑ) such that either γ(ȿ) = ȿ+ ȶ or ȿ = γ(ȿ) +ȶ. If γ(ȿ) = ȿ+ ȶ⟹ γ (ȿ) = γ(ȿ) +γ(ȶ) by cancellative ⟹ γ(ȶ) = 0 ⟹ ȶ∈ ker(γ) ⟹ γ(ȿ) = ȿ and hence γ=$1_{\mathbf{E(Ȑ)}}$, this shows E($_{Ȑ}$Ȑ) is indecomposable and uniform. Thus $_{Ȑ}$Ȑ is uniform. Now, let δ ∈ End(E($_{Ȑ}$Ȑ)), then by hypotheses either δ(1) ∈ Ȑ or δ(ȓ) = 1,for some ȓ ∈ Ȑ. If δ(1) ∈ Ȑ ⟹ δ(Ȑ) ⊆ Ȑ, if δ(ȓ) = 1, then $\delta|_{Ȑȓ}$:Ȑȓ ⟶ Ȑ is an isomorphism (because E($_{Ȑ}$Ȑ) is uniform and injective), δ is isomorphism on E($_{Ȑ}$Ȑ) and δ^{-1}(Ȑ) = Ȑȓ ⊆Ȑ, by Proposition(2.14) Ȑ is almost self-injective semimodule. ∇

In [3] the concept semiring of quotient is discussed and studied. From the previous proposition, the following result is obtained:

Corollary 2.16. Let $\mathcal{D}$ be semi-domain has injective hull and $\mathfrak{Q}$ its semiring of quotients, then $\mathcal{D}$ is almost self -injective if and only if for every q ∈ $\mathfrak{Q}$, either q or q^{-1} ∈ $\mathcal{D}$. This means $\mathcal{D}$ is almost self -injective if and only if $\mathcal{D}$ is valuation semi-domain.

Lemma 2.17. Suppose Ṃ is an indecomposable almost self-injective semimodule, if δ:Ụ⟶Ṃ, where Ụ ≤ Ṃ is a homomorphism which cannot be extended to an endomorphism of Ṃ, then there is a monic endomorphism θ of Ṃ such that θ is left inverse of δ on δ (Ụ), but θ is not invertible.

Proof: From definition there exists an Ȑ-homomorphism θ:Ṃ⟶Ṃ such that θδ(u) = u, for any u ∈Ụ. Hence θ is a left inverse of $\delta|_{\delta(Ụ)}$, since Ṃ is uniform by proposition(2.13), θ is monic. If θ is invertible, then θ^{-1} would be an extension of δ. ∇

Recall, an element r ∈ Ȑ is called nilpotent if there is positive integer n such that r^n = 0. An ideal $\mathcal{J}$ of Ȑ is called nil if each of its elements is nilpotent [10].

Proposition 2.18. Let Ȑ be a local semiring with Rad(Ȑ) nil ideal. If Ȑ is almost self-injective, then Ȑ is almost-injective.

Proof: Suppose Ȑ is almost self-injective, we have an Ȑ-homomorphism θ:Ǩ⟶Ȑ where Ǩ< $_{Ȑ}$Ȑ which cannot be extended to an endomorphism of $_{Ȑ}$Ȑ from Lemma(2.17), there is ϑ: $_{Ȑ}$Ȑ⟶ $_{Ȑ}$Ȑ such that ϑ(θ (ӽ)) = ӽ, ∀ ӽ∈ Ǩ ⟹ ϑ is not invertible. If it invertible, then θ(ӽ) = ϑ^{-1}(ӽ), ∀ ӽ∈ Ǩ ⟹ θ can be extended to ϑ^{-1} which is contrary with hypotheses, hence ϑ(1) = ɍ is not invertible ⟹ ɍ ∈ Rad(Ȑ) and ϑ is monic ⟹ if ϑ (ŧ) =ŧ ϑ(1) = ŧ ɍ = 0, then ŧ = 0 but this contradiction with hypotheses, hence Ȑ is self-injective. ∇

It is mentioned before that the symbol, ʃ denotes End(M̦), the semiring of endomorphisms of M̦. In the following, also, we mean by symbols $\subset$ proper subset and $\subseteq$ subset.

Proposition 2.19. Suppose M̦ is an indecomposable, semisubtractive, cancellative, almost self-injective semimodule, then $\forall \omega, \varphi \in$ ʃ ,we have:

(1) If $\ker(\omega) \subset \ker(\varphi)$, then ʃ$\varphi \subset$ ʃω .

(2) If $\ker(\omega) = \ker(\varphi)$, then ʃ$\varphi \subseteq$ ʃω or ʃ$\omega \subseteq$ ʃφ .

Proof: (1) Let $\omega, \varphi \in$ ʃ such that $\ker(\omega) \subset \ker(\varphi)$. Define ξ: ω(M̦)$\longrightarrow \varphi$ (M̦) by $\xi(\omega(m)) = \varphi(m)$ is well define, if $\omega(m) = \omega(n)$ for every m, n $\in$ M̦,by semisubtractive of M̦, $\exists$ş $\in$ M̦ such that either m = n+ ş or n= m+ ş if m = n+ ş $\Rightarrow \omega(m) = \omega(n) + \omega$(ş) by cancellative, ω (ş) = 0 $\Rightarrow$ ş$\in \ker(\omega) \subset \ker(\varphi) \Rightarrow$ ş$\in \ker(\varphi) \Rightarrow \varphi(m) = \varphi(n)$, the same result hold if n = m+ ş. ξ is not one to one(since $\ker(\omega) \subset \ker(\varphi)$), by hypothesis, ξ can be extended to M̦. Hence there is $\rho \in$ ʃ such that $\rho\big(\omega(m)\big) = \xi(\omega(m))$ $\forall m \in$M̦ $\Rightarrow \rho\big(\omega(m)\big) = \varphi(m) \Rightarrow$ ʃ$\varphi \subset$ ʃω. As for the second diagram, since M̦ is indecomposable, $\exists\sigma$: M̦ $\longrightarrow$ M̦ such that $\sigma\xi = 1_{M}$ this means ξ is one to one and this contradicts that ξ is not one to one). (2) Suppose $\ker(\omega) = \ker(\varphi)$, then ξ is one to one (in fact, for $\omega(m)$ and $\omega(n)$ in ω(M̦) such that $\xi(\omega(m)) = \xi(\omega(n)) \Rightarrow \varphi(m) = \varphi(n)$,by semisubtractive $\exists$ş $\in$ M̦ such that either m = n+ ş or n = m+ ş if m = n+ ş $\Rightarrow \varphi(m) = \varphi(n) + \varphi$(ş), by cancellative, we have ş$\in \ker(\varphi) = \ker(\omega) \Rightarrow \omega(n) = \omega(m)$. So either ξ can be extended to an endomorphism $\rho \in$ ʃ, or there exists $\gamma \in$ ʃ such that $\gamma\,\xi = 1_{\omega(M)} \Rightarrow \omega(m) = \gamma(\xi(\omega(m))) = \gamma(\varphi(m)) = \gamma\,\varphi(m)$ $\forall m \in$M̦. Thus ʃ$\omega \subseteq$ ʃφ , if $\gamma = \rho$ on $\omega(m)$, then ʃ$\varphi \subseteq$ ʃω. ∇

A semimodule M̦ is said to be uniserial if for any two subsemimodules of M̦, one of them contained in the other.

Corollary 2.20. If M̦ is uniserial, semisubtractive, cancellative, almost self-injective right semimodule, then ʃ is left uniserial.

Lemma 2.21 .Let M̦ be an indecomposable, semisubtractive, cancellative almost self-injective semimodule, then the left ideal Ӥ of ʃ = End(M̦) generated by non-isomorphic monomorphisms in ʃ is two sided ideal.

Proof: Let $\varphi \in$ʃ and γ be non-isomorphism with $\ker(\gamma) = 0$, it is enough to show that $\gamma\varphi \in$ Ӥ. If $\ker(\gamma\varphi) \neq 0$, from proposition(2.19) ($\ker(\gamma) \subset \ker(\gamma\varphi)$) implies that ʃ$\gamma\varphi \subset$ ʃγ) then, $\gamma\varphi \in$ Ӥ. If $\ker(\gamma\varphi) = 0 = \ker(1_{M})$, by proposition(2.19), then ʃ$\gamma\varphi \subseteq$ ʃ1_{M} if $\gamma\varphi$ is isomorphism implies that γ is onto, a contradiction. Thus $\gamma\varphi \in$ Ӥ. ∇

Proposition 2.22. The endomorphism semiring of indecomposable, semisubtractive, cancellative almost self-injective semimodule is local.

Proof: Since M̦ is indecomposable almost self-injective semimodule, then it is uniform by proposition(2.13), let $\tilde{\Omega}$ be the set of all non-isomorphism monomorphism in ʃ. If $\tilde{\Omega} = \emptyset$, then $\psi \in$ ʃ is an isomorphism if and only if $\ker(\psi) = 0$, let U(ʃ) be the monoid of invertible elements of ʃ and $\alpha + \beta \in$ U(ʃ), since M̦ is uniform, either α or β is monomorphism, this mean either α or β is an isomorphism. Therefore ʃ is local. Now suppose $\tilde{\Omega} \neq \emptyset$, let Ӥ= $\sum_{\gamma \in \tilde{\Omega}}$ ʃγ, by proposition (2.19)ʃ \U(ʃ)$\subset$Ӥ, we must show that $\propto \in$ Ӥ is not invertible, $\propto = \sum_{i=1}^{n} f_i \gamma_i$ where $\gamma_i \in \tilde{\Omega}$ and $f_i \in$ʃ. By Proposition (2.19), ʃγ_1, ʃγ_2,…, ʃγ_n are linearly ordered by inclusion relation. Hence $\propto = f\gamma_n$ for some f $\in$ʃ, if$\propto$ is invertible, then γ_n is left invertible, since ʃ has no nontrivial idempotent, γ_n is invertible and this contradiction with $\gamma_n \in \tilde{\Omega}$. Thus ʃ \U(ʃ)=Ӥ. Since Ӥ is two sided ideal of ʃ by Lemma(2.21), we have ʃ is local. ∇

Proposition 2.23. Let M̦ be almost self-injective semimodule, then:

(1) If $\mathcal{S}$ is local and M is indecomposable, then M is uniform.

(2) If M is uniform, then $Z(\mathcal{S}) \subset Rad(\mathcal{S})$.

Proof: (1) Suppose ¥∩Ӿ = 0, where Ӿ and ¥ are subsemimodules of M (since M is almost self-injective semimodule and indecomposable) by Proposition(2. 12), then M is π-injective and hence there exist $\omega, \varphi \in \mathcal{S}$, such that Ӿ ⊆ $\ker\omega$ and ¥ ⊆ $\ker\varphi$ and $\omega + \varphi = 1_M$. Since $\mathcal{S}$ is local either $\omega \in Rad(\mathcal{S})$ or $\varphi \in Rad(\mathcal{S})$, if $\omega \in Rad(\mathcal{S})$, then φ is invertible implies $\ker\varphi = 0$, since ¥ ⊆ $\ker\varphi$, hence ¥ = 0.

(2) Let $\varphi \in Z(\mathcal{S})$ and $0 \neq \omega \in \mathcal{S}$, $ann(\varphi)$ is essential in $\mathcal{S}$, there exists $f \in \mathcal{S}$ such that $0 \neq f\omega \in ann(\varphi) \Rightarrow \varphi(f\omega) = 0$. If $\ker(\varphi) = 0$, then φ is monic, then if $\varphi(f\omega) = 0 \Rightarrow f\omega = 0$ contradiction. Then $\ker(\varphi) \neq 0$, since M is uniform , then $\ker(\varphi)$ essential in M . Now, since $\ker(\varphi) \cap \ker(1+\omega\varphi) = 0$, in fact,if $x \in \ker(\varphi) \cap \ker(1+\omega\varphi) \Rightarrow \varphi(x) = 0$ and $(1+\omega\varphi)(x) = 0 \Rightarrow 1(x) = 0$, we have $\ker(1+\omega\varphi) = 0 = \ker(1_M)$ by proposition(2.19) $\Rightarrow$ there is $h \in \mathcal{S}$ such that $1_M = h(1+\omega\varphi) \Rightarrow \varphi \in Rad(\mathcal{S})$ by Lemma(1.14). ∇

Proposition 2.24. Suppose M is an indecomposable almost self-injective semimodules and $\omega, \varphi \in \mathcal{S}$,

(1) If $\omega(M)$ embeds in $\varphi(M)$, then $\mathcal{S}\omega$ is a homomorphic image of $\mathcal{S}\varphi$.

(2) If $\omega(M) \cong \varphi(M)$, then $\mathcal{S}\omega \cong \mathcal{S}\varphi$.

Proof: (1) Let $\xi: \omega(M) \longrightarrow \varphi(M)$ be a monomorphism, and $i_1: \omega(M) \longrightarrow M$, $i_2: \varphi(M) \longrightarrow M$ be the inclusion maps, since M is almost self-injective, either there exists $\rho \in \mathcal{S}$ such that $\rho\, i_1 = i_2\, \xi$. Define $\mu: \mathcal{S}\varphi \longrightarrow \mathcal{S}\omega$ by $\mu(f\varphi) = f\rho\,\omega$, $\forall f \in \mathcal{S}$. μ is well defined (if, $f_1\varphi = f_2\varphi, \forall m \in M \Rightarrow f_1\rho\,\omega(m) = f_1\xi\omega(m) = f_1\varphi(m) = f_2\varphi(m) = f_2\xi\omega(m) = f_2\rho\,\omega(m)$, $\forall m \in M$, then $f_1\rho\,\omega(m) = f_2\rho\,\omega(m)$and hence μ is well defined. Now to show that $\ker(\omega) = \ker(\rho\,\omega)$, let $x \in \ker(\rho\,\omega) \Rightarrow \rho\omega(x) = 0$, $\xi(\omega(x)) = 0 \Rightarrow \omega(x) = 0$ because ξ is monic. It is clear that that $\ker(\omega) \subset \ker(\rho\,\omega)$, hence $\ker(\omega) = \ker(\rho\,\omega)$ by proposition (2.19), $\mathcal{S}\omega \subset \mathcal{S}\rho$, so $\omega = f\rho\,\omega$ for some $f \in \mathcal{S}$, then $\omega = f\rho\,\omega = \mu(f\varphi) \Rightarrow \mathcal{S}\omega \subset \mathcal{S}f\rho\,\omega = \mu(\mathcal{S}\varphi)$ this means μ is $\mathcal{S}$-epimorphism. Or $\exists \beta \in \mathcal{S}$ such that $\beta i_2\xi = 1_{\omega(M)} \Rightarrow \omega(m) = \beta i_2\xi(\omega(m)) = \beta\, i_2(\varphi(m)) = \beta\varphi(m)$, for every m in $M \Rightarrow \mathcal{S}\omega \subseteq \mathcal{S}\varphi$, then μ is $\mathcal{S}$-epimorphism.

(2) Let $\xi: \omega(M) \longrightarrow \varphi(M)$ be an Ȑ-isomorphism and $i_1: \omega(M) \longrightarrow M$, $i_2: \varphi(M) \longrightarrow M$ be the inclusion maps, since M is almost self-injective, either there exists $\rho \in \mathcal{S}$ such that $\rho\, i_1 = i_2\, \xi$, or $\rho\, i_2\, \xi = 1_{\omega(M)}$. From (1) μ is $\mathcal{S}$- epimorphism, to show μ is monomorphism, let $f(\varphi)) \in \ker(\mu)$, thus $\mu(f(\varphi)) = 0$ so $f\rho\,\omega = 0$ since $\rho\,\omega(M) = \varphi(M)$, $f\rho\,\omega(M) = f\varphi(M)$ hence $f\varphi(M) = 0$ implies $f\varphi = 0$. ∇

In [6] the set $W(\mathcal{S}) = \{\varphi \in \mathcal{S} / \ker\varphi$ is essential in $M\}$ is two sided ideal of $\mathcal{S}$.

Proposition 2.25. If M is cancellative, semisubtractive almost self- injective semimodule, then:

(1) $W(\mathcal{S}) \subseteq Rad(\mathcal{S})$.

(2) If $\mathcal{S}$ is local, then $Rad(\mathcal{S}) = \{\vartheta \in \mathcal{S} : \ker(\vartheta) \neq 0\}$.

Proof: (1)Let $\alpha \in W(\mathcal{S})$ and let $\theta \in \mathcal{S}$ since $\ker(\alpha) \cap \ker(1+\theta\alpha) = 0$ and $\ker(\alpha) \leq_e M$, then $\ker(1+\theta\alpha) = 0 = \ker(1_M)$ by proposition (2.19) $\mathcal{S} \subseteq \mathcal{S}(1+\theta\alpha)$, since $1 \in \mathcal{S} \Rightarrow 1 = f(1+\theta\alpha)$ for some f in $\mathcal{S}$ and hence $\alpha \in Rad(\mathcal{S})$. (2)Since $\mathcal{S}$ is local, $\mathcal{S}\alpha \neq \mathcal{S}$ for any $\alpha \in Rad(\mathcal{S})$, let $\alpha \in Rad(\mathcal{S})$ to show that $\ker(\alpha) \neq 0$. Suppose that $\ker(\alpha) = 0$, define $\psi: \alpha(M) \longrightarrow M$ by $\psi(\alpha(ŧ)) = ŧ$, ψ is well defined, for any ŧ, ë in M, assume that $\alpha(ŧ) = \alpha(ë)$ by semisubtractive, there is ȿ in M such that, either ŧ = ë+ ȿ or ë = ŧ+ ȿ, we get $\alpha(ŧ) = \alpha(ë) + \alpha(ȿ)$ by cancellative $\alpha(ȿ) = 0 \Rightarrow$ ȿ $\in \ker(\alpha) \Rightarrow$ ȿ $= 0$, then ŧ = ë, the same way for ë = ŧ+ ȿ. Since M is almost self-injective semimodule, either there is endomorphism η of M such that $\eta i = \psi$, then $\eta\alpha = \eta i\alpha = \psi\alpha = 1_M$ so $\ker(\eta\alpha) = \ker(1_M)$ by proposition (2.19) $\mathcal{S}\eta\alpha \subseteq \mathcal{S}\alpha$ and hence $\mathcal{S}\alpha = \mathcal{S}$ this contradiction, then

Rad(ʆ) ⊆{ϑ ∈ ʆ | ker (ϑ) ≠0}, or ∃σ ∈ ʆ such that σ ψ = $1_{\alpha(M)}$ ⟹ α(M) =σ ψ(α(M)) =σ(M), this means ʆα =ʆ this contradicts the assumption. Conversely, let α ∈{ϑ ∈ ʆ | ker (ϑ) ≠0}, since ʆ is local, then Rad(ʆ) = {α ∈ ʆ: ʆα ≠ ʆ }. suppose that ʆα = ʆ⟹ βα = 1_M for some β ∈ ʆ , so ker(βα)= ker (1_M) = 0, then ker(α) ⊆ ker(βα), then ker(α) = 0, a contradiction.∇

Proposition2.26. Let M be cancellative, semisubtractive almost self-injective semimodule with ʆ = End(M). If α(M) is a simple right R̀-semimodule, then ʆα is a simple left ʆ-semimodule, where α ∈ ʆ.

Proof: Let D̂ be nonzero subsemimodule of ʆα and 0 ≠ βα ∈ D̂, then ʆβα ⊂D̂. Suppose ker(β) ∩ α(M) ≠ 0. Since α(M) is simple and ker(β) ∩ α(M) ⊂ α(M)⟹ ker(β) ∩ α(M)= α(M)⟹ α(M) ⊂ ker(β), so βα(M) = 0⟹ βα =0, a contradiction so ker(β) ∩ α(M) =0⟹ ker(βα) = ker(α), hence by proposition(2.19) ʆβα ⊆ ʆα ⟹ ʆβα ⊂D̂ ⊂ʆα ⟹ D̂ =ʆα. ∇

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